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REINFORCED CONCRETE CELLULAR ORTHOTROPIC SLABS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled REINFORCED CONCRETE CELLULAR ORTHOTROPIC SLABS submitted by Reginald George Quinton in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The results of a study of the behaviour of reinforced concrete cellular orthotropic slabs is presented together with guidelines for their design. The study was restricted to cellular flat plates supported on non-deflecting columns with no edge beams or column capitals. The variables considered were the stiffness properties of the slab, the span ratio of the panels, and the exterior column stiffnesses. An analysis was performed by the method of finite differences.

Results of the study show that cellular orthotropic slabs behave similar to solid isotropic slabs when the span ratio or the exterior column stiffness is varied. Varying the stiffness properties of the slab has little effect on the total moment at any given section but has an effect on the distribution of moments across the section. Slab deflections are greatly effected by changes in slab stiffness properties.

Methods for approximating the distribution of moments across a section and for calculating deflections for a cellular orthotropic slab are presented.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

A_x, A_y	=	areas of the cross-sections perpendicular to the x and y directions, respectively.
b_x, b_y	=	width of a cell unit perpendicular to the x and y directions, respectively.
b_{ix}, b_{iy}	=	width of a cell perpendicular to the x and y directions, respectively.
C_x, C_y	=	flexural rigidity or stiffness factors for a cellular orthotropic slab.
C_{xy}, C_{yx}	=	torsional rigidity or stiffness factors for a cellular orthotropic slab.
d_x	=	$b_x - b_{ix}$
d_y	=	$b_y - b_{iy}$
D	=	$\frac{Eh^3}{(1-\nu^2)12}$ = flexural rigidity or stiffness of a solid isotropic slab.
D_x, D_y, D_{1x}, D_{1y}, D_{xy}, D_{yx}	=	rigidity or stiffness factors of an orthotropic plate.
E	=	modulus of elasticity.
E_{cc}	=	modulus of elasticity for column concrete.
E_{cs}	=	modulus of elasticity for slab concrete.
E'_x, E'_y, E'', G	=	characteristic elastic properties of an orthotropic material.
F	=	modification factor for column strip moments for variations in the D_x/H ratio.
h	=	height or thickness of slab.
h_i	=	height of a cell.
ht	=	height of column.
hx, hy	=	distance between finite difference grid points in the x and y directions, respectively.

H = $(D_{1x} + D_{1y} + 2D_{xy} + 2D_{yx})/2$

I_c = moment of inertia of column.

I_s = moment of inertia of the slab between the centers of the spans either side of column.

I_x, I_y = moments of inertia of the cross-sections perpendicular to x and y directions, respectively.

J_x, J_y = torsional constants for the cross-sections perpendicular to x and y directions, respectively.

k_c = factor reflecting the effect of support conditions and variation in cross-section on the flexural stiffness of a column.

k_s = factor reflecting the effect of support conditions, variation in cross-section and shape of panel on the flexural stiffness of a slab.

K' = $\dfrac{\Sigma(k_c E_{cc} I_c/ht)}{\Sigma(k_s E_{cs} I_s/l_x)}$ = relative flexural stiffness of the columns above and below the slab to the flexural stiffness of the slab.

l_x = length of span for which the moments are being determined.

l_y = length of span transverse to l_x.

Mo = total static moment of a panel.

M_x, M_y = bending moments per unit length of cross-sections perpendicular to x and y directions, respectively.

M_{xy}, M_{yx} = twisting moments per unit length of cross-sections in rectangular coordinates.

q = load per unit area.

Q_x, Q_y = shearing forces on cross-sections perpendicular to x and y directions, respectively.

span ratio	=	l_x/l_y = longitudinal to transverse span.
t	=	$h - h_i$
w	=	vertical displacement or deflections of the slab.
x,y	=	rectangular coordinate directions.
z	=	distance from the neutral axis of any point on the cross-section measured perpendicular to the neutral axis.
σ_x, σ_y	=	normal stress components in the x and y directions, respectively.
τ_{xy}, τ_{yx}	=	shearing stress components in rectangular coordinates.
ε_x, ε_y	=	unit elongation or strain in x and y directions, respectively.
γ_{xy}, γ_{yx}	=	shearing strain in rectangular coordinates.
δ_x	=	b_{ix}/b_x
δ_y	=	b_{iy}/b_y
λ	=	h_i/h
ν	=	Poisson's ratio

CHAPTER I

INTRODUCTION

1.1 Introductory Remarks

A reinforced concrete cellular orthotropic slab is a concrete slab which has an orthogonal network of cavities or cells which may or may not extend to the top or bottom surface of the slab. The shape of the cells or cavities is arbitrary but the dimensions of each cell are small in comparison to the clear span of the slab. Because this type of slab has a definite reduction in dead weight over a solid slab with equivalent shear and bending moment capacities, it can be very useful in slab construction, especially in structures with long spans where the dead weight is the major part of the total load.

Generally, the use of this type of slab construction has been restricted to slabs with equal stiffness properties in the two coordinate directions (square waffle slabs) where the slab behaviour is similar to that of a solid isotropic slab. To date there has been little to guide the designer in proportioning moments and predicting behaviour if the cells are such as to cause different relative stiffnesses in the coordinate directions. Therefore, the purpose of this research was to study the behaviour of these slabs with different stiffness properties in the coordinate directions, to determine how they differ from solid isotropic slabs, and from this information to suggest guidelines for a rational design of cellular slabs.

1.2 Review of Related Literature

In a review of English language publications a fairly large number of articles were found on orthotropic plates. However, practically all these articles either were dealing with orthotropic steel plates or were restricted to special boundary conditions not usually found in concrete slab construction.

In 1964 F. Pfeffer(1) published an article entitled "Stahlbeton-Zellworke" (Reinforced Concrete Cellular Structures) which deals with the type of cellular slab construction being considered in this study. Pfeffer derived expressions for the stiffness constants for various types of cell units and presented an example of an analysis for a cellular slab supported on a very irregular boundary using a finite difference technique. However, he did not attempt to determine the effect of variations in the stiffness properties on slab behaviour and to draw general conclusions which would give direction in the design of cellular slabs.

1.3 Scope of Study

This study was restricted to cellular flat plates with no edge beams or column capitals. The cross-sections of the slab could be different in the orthogonal directions, but for any cross-section the cell units were of uniform size across the entire section. The loading was considered uniform over the entire slab.

1.4 Outline of Procedure

The analysis of a cellular slab involved two parts; firstly the determination of the stiffness or rigidity properties and secondly the determination of the deflections and bending moments. Expressions for the stiffness factors for the most common types of cellular slabs were derived using the procedure presented by Pfeffer[1]. To determine the deflections and moments in the slab a finite difference technique was used. A computer program was written which, given the properties of the slab, calculated deflections and moments. With this program the properties of the slab were varied independently and the individual effect of each property was studied. From these results guidelines for the behaviour and design of cellular slabs were established.

CHAPTER II

CELLULAR ORTHOTROPIC THEORY

2.1 Classical Orthotropic Theory

The theory of orthotropic elasticity is given in detail by Hearman[2] and to a lesser extent by Timoshenko[3]. The bending and twisting moment equations for orthotropic plates are derived in APPENDIX A and are as follows:

$$
\begin{aligned}
M_x &= \int_{-h/2}^{h/2} \sigma_x z dz = -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_{1x} \frac{\partial^2 w}{\partial y^2}\right) \\
M_y &= \int_{-h/2}^{h/2} \sigma_y z dz = -\left(D_y \frac{\partial^2 w}{\partial y^2} + D_{1y} \frac{\partial^2 w}{\partial x^2}\right) \\
M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} z dz = -\ 2D_{xy} \frac{\partial^2 w}{\partial x \partial y} \\
M_{yx} &= \int_{-h/2}^{h/2} \tau_{yx} z dz = -\ 2D_{yx} \frac{\partial^2 w}{\partial x \partial y}
\end{aligned}
\tag{2.1}
$$

where D_x, D_y, D_{1x}, and D_{yx} are the stiffness or rigidity factors for an orthotropic plate. These stiffness properties, analogous to the stiffness of a beam, can be effected by two factors, the elastic properties of the material and the geometric properties of the cross-section.

The plate equilibrium equation given by Timoshenko[3] is:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \qquad (2.2)$$

By substituting equations (2.1) into (2.2) the differential equation for orthotropic plates is obtained:

$$D_x \frac{\partial^4 w}{\partial x^4} + (D_{1x} + D_{1y} + 2D_{xy} + 2D_{yx}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \qquad (2.3)$$

Using the notation:

$$H = (D_{1x} + D_{1y} + 2D_{xy} + 2D_{yx})/2$$

the equation is changed to its standard form:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \qquad (2.4)$$

2.2 Orthotropic Theory Applied to Cellular Slabs

A cellular slab differs from the classical orthotropic plate in that the elastic properties of its material can be assumed isotropic with a constant modulus of elasticity. A cellular slab gets its orthotropic character from the geometry of its cross-sections. Because of the

isotropic nature of the material the stress-displacement relationships for an isotropic slab can be used for an orthotropic slab. They are as follows:

$$\sigma_x = \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

$$\sigma_y = \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \qquad (2.5)$$

$$\tau_{xy} = \tau_{yx} = \frac{Ez}{1+\nu} \cdot \frac{\partial^2 w}{\partial x \partial y}$$

The basic equations for the bending and twisting moments of a plate are given by the following integrals:

$$M_x = \int_{-h/2}^{h/2} \sigma_x \, zdz$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y \, zdz \qquad (2.6)$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, zdz$$

$$M_{yx} = \int_{-h/2}^{h/2} \tau_{yx} \, zdz$$

If the cellular cross-section is known, equations (2.5) can be substituted into equations (2.6) and the integration performed. By comparing these equations with equations (2.1) the stiffness factors, D_x, D_y, D_{1x}, D_{1y}, D_{xy}, and D_{yx}, can be determined. This is the approach used by Pfeffer [1]. Derivation of stiffness factors by this method

is shown in detail for a prismatic cellular slab in the following section. Derivations for other types of cellular slabs are shown in APPENDIX B.

2.3 Stiffness Factors for Prismatic Cell Units

The type of cellular slab considered here consists of prismatic hollow bodies as shown in FIGURE 2.1. The characteristics of the cross-sections can be given by the three expressions:

$$\delta_x = \frac{b_{ix}}{b_x} \quad , \quad \delta_y = \frac{b_{iy}}{b_y} \quad , \quad \lambda = \frac{h_i}{h} \tag{2.7}$$

Using the expressions for σ_x, σ_y, τ_{xy}, and τ_{yx} given by equations (2.5) the equation for the bending moment M_x can be expressed as:

$$M_x = \frac{1}{b_x} \left(b_x \int_{-h/2}^{h/2} \sigma_x \, zdz \; - \; b_{ix} \int_{-h_i/2}^{h_i/2} \sigma_x zdz \right)$$

$$= \frac{E\ h^3}{(1-\nu^2)12} (1 - \delta_x \lambda^3) \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

Similarly:

$$M_y = \frac{E\ h^3}{(1-\nu^2)12} (1 - \delta_y \lambda^3) \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \tag{2.8}$$

$$M_{xy} = \frac{1}{b_x} \left(b_x \int_{-h/2}^{h/2} \tau_{xy} \, zdz - b_{ix} \int_{-h_i/2}^{h_i/2} \tau_{xy} \, zdz \right)$$

$$= \frac{E\ h^3}{(1+\nu)12} (1 - \delta_x \lambda^3) \frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xy} = \frac{E\ h^3}{(1+\nu)12}(1-\delta_y\lambda^3)\ \frac{\partial^2 w}{\partial x \partial y}$$

Comparing equations (2.8) with equations (2.1) the stiffness factors are as follows:

$$\begin{aligned}
D_x &= \frac{E\ h^3}{(1-\nu^2)12}\ (1-\delta_x\lambda^3) &&= D.C_x \\
D_y &= \frac{E\ h^3}{(1-\nu^2)12}\ (1-\delta_y\lambda^3) &&= D.C_y \\
D_{1x} &= \frac{\nu E\ h^3}{(1-\nu^2)12}\ (1-\delta_x\lambda^3) &&= \nu.D.C_x \\
D_{1y} &= \frac{\nu E\ h^3}{(1-\nu^2)12}\ (1-\delta_y\lambda^3) &&= \nu.D.C_y \\
2D_{xy} &= \frac{E\ h^3}{(1+\nu)12}\ (1-\delta_x\lambda^3) &&= (1-\nu)\ D.C_{xy} \\
2D_{yx} &= \frac{E\ h^3}{(1+\nu)12}\ (1-\delta_y\lambda^3) &&= (1-\nu)\ D.C_{yx}
\end{aligned} \tag{2.9}$$

where

$$D = \frac{E\ h^3}{(1-\nu^2)12}$$

is the stiffness of a solid isotropic slab and

$$C_x = C_{xy} = 1 - \delta_x\lambda^3$$

$$C_y = C_{yx} = 1 - \delta_y\lambda^3$$

2.4 Stiffness Factors for Other Cell Units

For all types of cellular slabs the stiffness factors can be expressed in terms of the four factors, C_x, C_y, C_{xy}, and C_{yx}, where:

$$D_x = D.C_x \qquad D_y = D.C_y$$

$$D_{1x} = \nu.D.C_x \qquad D_{1y} = \nu.D.C_y \tag{2.11}$$

$$2D_{xy} = (1-\nu)\ D.C_{xy} \qquad 2D_{yx} = (1-\nu)\ D.C_{yx}$$

and

$$H = D(\nu C_x + \nu C_y + (1-\nu)\ C_{xy} + (1-\nu)\ C_{yx})/2 \tag{2.12}$$

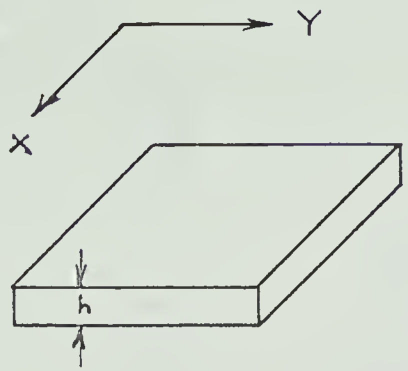
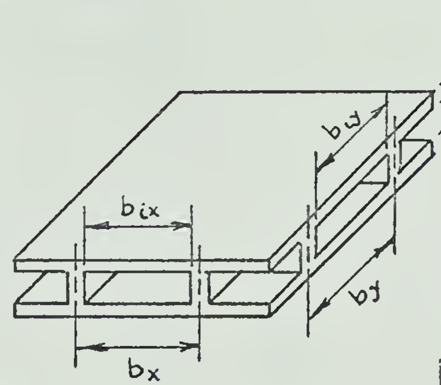
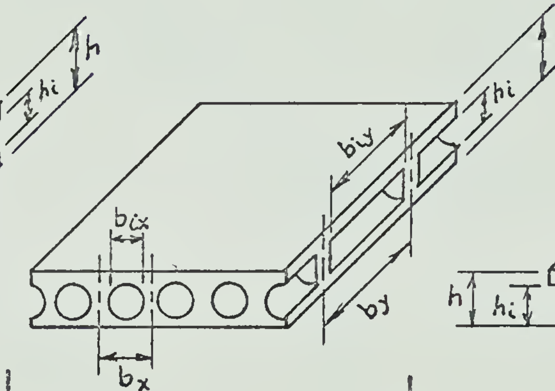
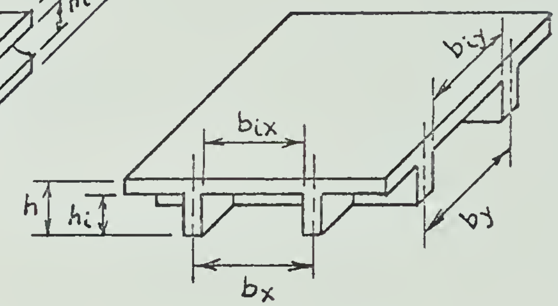
The factors, C_x, C_y, C_{xy}, and C_{yx}, are always less than or equal to one. When they are all equal to one, the slab is solid and isotropic. TABLE 2.1 gives the stiffness factors for the most common types of cellular slabs.

2.5 Weight Reduction Factors

The big advantage in the use of cellular slabs is the saving in dead weight. By dividing the volume of the cell unit removed by the volume of a solid cell unit a weight reduction factor can be obtained. These weight reduction factors are given in TABLE 2.1.

TABLE 2.1

STIFFNESS FACTORS

TYPE OF CELLULAR SLAB				
C_x	1	$1-\delta_x\lambda^3$	$1-\frac{3\pi}{16}\delta_x\lambda^3$	$\frac{1}{1-\delta_x\lambda}(1-\delta_x\lambda+6\delta_x\lambda^2-4\delta_x\lambda^3+\delta_x^2\lambda^4)$
C_y	1	$1-\delta_y\lambda^3$	$1-\delta_y\lambda^3(\frac{2}{3}\delta_x)^{\frac{3}{2}}$	$\frac{1}{1-\delta_y\lambda}(1-\delta_y\lambda+6\delta_y\lambda^2-4\delta_y\lambda^3+\delta_y^2\lambda^4)$
C_{xy}	1	$1-\delta_x\lambda^3$	$1-\frac{3\pi}{16}\delta_x\lambda^3$	$(1-\lambda)^3+a_x^{\dagger}\lambda(1-\delta_x)$
C_{yx}	1	$1-\delta_y\lambda^3$	$1-\delta_y\lambda^3(\frac{2}{3}\delta_x)^{\frac{3}{2}}$	$(1-\lambda)^3+a_y^{\dagger}\lambda(1-\delta_y)$
WEIGHT REDUCTION FACTOR	0	$\lambda\delta_x\delta_y$	$\frac{\pi}{4}\lambda\delta_x\delta_y$	$\lambda\delta_x\delta_y$

$\dagger$ $a_x=2(\frac{d_x}{h})^2.(1-0.630\frac{d_x}{h}+0.052(\frac{d_x}{h})^5)$ $\quad d_x=b_x-b_{ix}$

$a_y=2(\frac{d_y}{h})^2.(1-0.630\frac{d_y}{h}+0.052(\frac{d_y}{h})^5)$ $\quad d_y=b_y-b_{iy}$

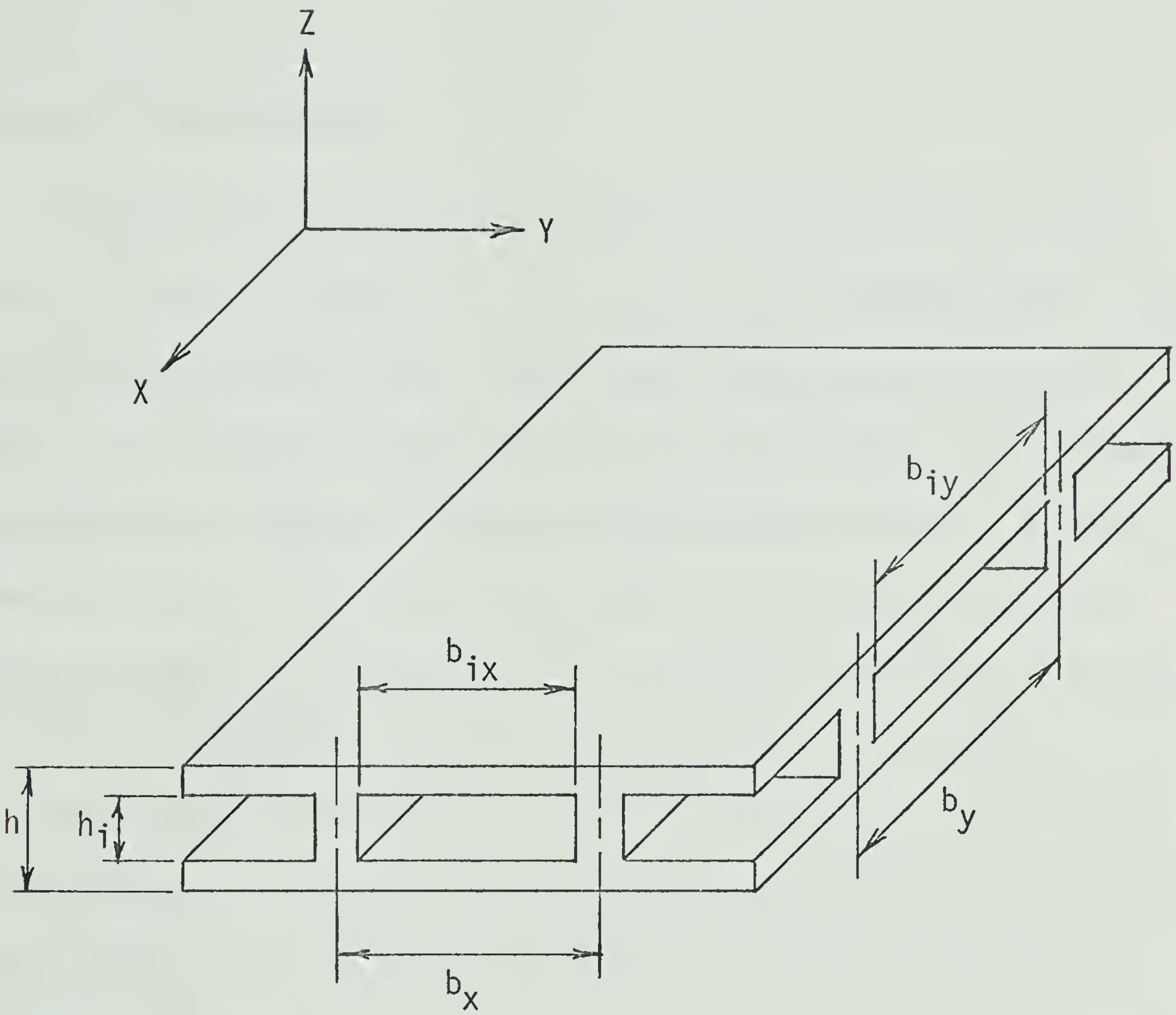


FIGURE 2.1 PRISMATIC CELL UNITS

CHAPTER III

METHOD OF ANALYSIS

3.1 Type of Slab Analysed

The slab analysed was a nine panel flat plate with quarter symmetry as shown in FIGURE 3.1. This gave three different panel types, interior, exterior and corner. There were no edge beams or column capitals. The loading of the slab was assumed uniform. The columns were considered non-deflecting and concentrated at a point. A study by Simmonds and Seiss[4] has indicated that for relatively uniform loadings the effect of varying the column bending stiffness for columns across which the slab is continuous is negligible. Therefore, in this study the bending stiffness of columns in the directions where the slab was continuous was considered constant and equal to infinity, but in the directions where the slab was discontinuous the column stiffness was considered a variable.

3.2 Finite Difference Method

The method of analysis was to replace, using a finite difference technique, the governing differential equation (2.4) with a set of simultaneous linear equation (see APPENDIX C). The slab was divided into eight divisions per panel in each direction giving a total of 169 equations.

A Gauss-Jordan elimination technique was used to solve the equations for deflections and these deflections were used to calculate bending moments.

Results were obtained using a computer program which given the input of the span ratio, the exterior column stiffness, and the stiffness properties of the slab gave an output of deflections and bending moments. This program was run on an IBM 360 MOD/67 computer.

A statics check was performed on the bending moments calculated by the analysis comparing the total moment in a span with the theoretical static moment. In all cases the moment from the analysis was within 2% of the theoretical value and that was considered satisfactory for this study.

3.3 Choice of Parameters

3.3.1 Slab Stiffness Parameters

The slab stiffness factors, D_x, D_y, and H, given in CHAPTER II, can be expressed as the dimensionless ratios, D_x/D, D_y/D, and H/D, where D is the stiffness of a solid isotropic slab of the same thickness. However, a more convenient form of expressing these ratios for this study was found to be in terms of D_x/D, D_x/D_y, and D_x/H.

The bending stiffnesses, D_x and D_y, of a slab can not be varied without effecting the twisting stiffness and the value of H. However, the relationship between D_x, D_y, and H varied with the type of cellular slab. An approximate relationship given by Timoshenko[3] is $H = \sqrt{D_x D_y}$

or in another form $D_x/H = \sqrt{D_x/D_y}$. This approximation was first used by M.T. Huber and is commonly referred to as Huber's approximation. While the effect of variable D_x/D and D_x/D_y ratios was being studied, D_x/H was equal to $\sqrt{D_x/D_y}$. Later the D_x/H ratio was varied with D_x/D and D_x/D_y constant.

After studying the stiffness constants for cellular slabs given in TABLE 2.1 and considering the limiting cases which would be practical to build, ranges of values for the stiffness factors were established. For some factors the range was set beyond practical values to show their effect in very extreme cases. The ratio D_x/D was considered to vary from 0.1 to 1.0, D_x/H from 1.0 to 20.0, and D_x/D_y from 0.25 to 4.0.

3.3.2 Other Parameters

Besides the stiffness properties of the slab the other parameters which were considered were the span ratio, the ratio of exterior column stiffness to slab stiffness, K', and Poisson's ratio, ν. The span ratio was allowed to vary from 0.5 to 2.0 which are the limits of the proposed Reinforced Concrete Building Code ACI 318-71[5]. While the span ratio and the stiffness properties were varied the ratio of the exterior column to slab stiffness was held constant at K' = 25, corresponding to a condition of very stiff exterior columns. Later K' was varied with values from 0 to 25. The definition used for K' was the same as given in the current draft of the proposed ACI 318-71 Code. The actual slab stiffness, D_x or D_y, not the solid slab stiffness, D,

was used as the slab stiffness to which the column stiffness was compared. In cellular slabs the concrete can expand laterally and the Poisson's ratio effect cannot greatly influence the stresses. Therefore, Poisson's ratio, ν, was assumed equal to zero for this study.

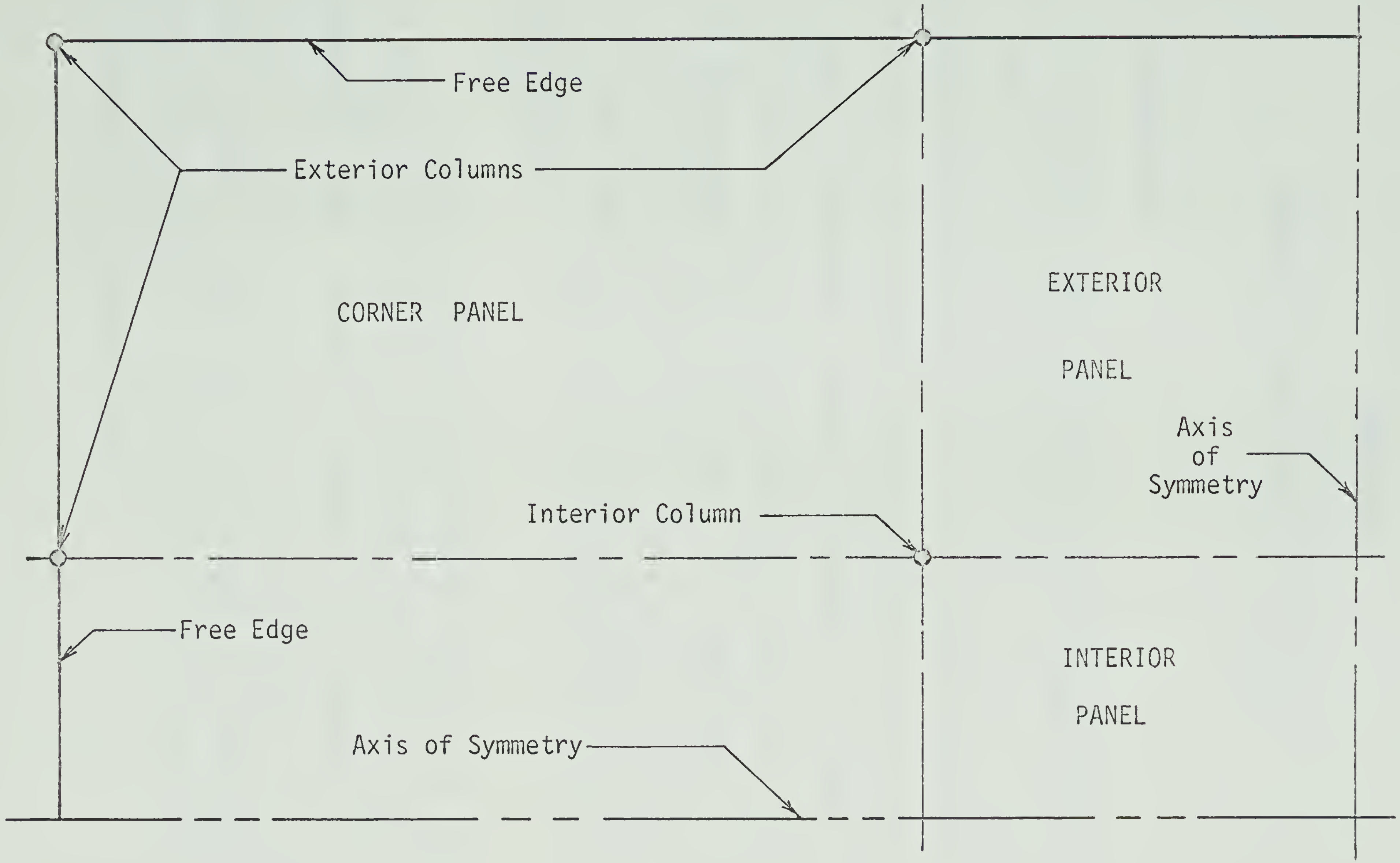


FIGURE 3.1 PLAN OF SLAB ANALYSED

CHAPTER IV

PRESENTATION AND DISCUSSION OF RESULTS

4.1 Introduction

The total static moment, Mo, in a slab panel depends only on the clear span between columns and the load, and does not depend on the stiffness properties of the slab. In the proposed Reinforced Concrete Building Code, ACI 318-71[5] this total moment is first split between the negative and positive sections of the panel. The moment assigned to each section is then proportioned between column and middle strips and is assumed constant across each strip. The current draft of proposed ACI 318-71 defines the column strip as one quarter of the short span either side of the column and the middle strip as the strip between two column strips.

In order that they may be helpful in design, the results of this study are presented in a form which can be easily related to the proposed ACI Code. The amount of moment in the negative and positive sections is shown as a percentage of the total moment, Mo, and the amount of moment in the column strip is shown as a percentage of the moment at that section. The definitions of column and middle strips are the same as given by ACI 318-71.

4.2 Co-ordinate System

In the co-ordinate system used to present the results the X-span is always the span in which the moments are being determined or the longitudinal span and the Y-span is always the transverse span. Ratios such as D_x/D_y and the span ratio change when the span being considered changes. For example a panel which has a span ratio of 2.0 when the long span moments are being considered has a span ratio of 0.5 when the short span moments are considered.

4.3 Variable D_x/D Ratio

Decreasing the D_x/D ratio while D_x/D_y is constant and $D_x/H = \sqrt{D_x/D_y}$ (Huber's approximation) weakens the slab proportionally in both bending and twisting stiffnesses and is analogous to decreasing the thickness of a solid isotropic slab. All the stiffness properties of the slab decrease and the curvatures and deflections increase, but the moments, which are stiffnesses multiplied by curvatures, remain unchanged.

These conclusions were confirmed by the results of this study which showed that when D_x/D was varied with D_x/D_y constant and $D_x/H = \sqrt{D_x/D_y}$ the moments in the slab remained constant while the deflections increased directly proportional to the inverse to D_x/D. Therefore, in the design of a cellular orthotropic slab the D_x/D ratio need only be considered when checking deflections.

4.4 Variable D_x/D_y Ratio

4.4.1 D_x/D_y - Span Ratio Analogy

If $D_x/D_y = 1.0$ and $D_x/H = \sqrt{D_x/D_y} = 1.0$, an isotropic condition exists with the stiffness properties equal, $D_x = D_y = H$. If D_x/D_y is not equal to 1.0, the slab is stiffer in one direction than in the other and the curvature in the weak direction increases relative to the curvature in the strong direction. If the span ratio is not equal to 1.0 the curvature in the long span increases relative to the curvature in the short span. Therefore, stiffening the slab in the longitudinal direction has an effect similar to increasing the span in the transverse direction, and increasing D_x/D_y can be thought of as analogous to decreasing the span ratio.

4.4.2 Negative-Positive Moment Split

FIGURES 4.1, 4.2, 4.3 and 4.4 give the negative-positive moment split for a variable D_x/D_y ratio for interior, exterior and corner panels, respectively. For comparison the design values for an interior panel of a solid isotropic slab given by the Reinforced Concrete Building Code, ACI 318-63[6] and the proposed ACI 318-71 Code[5] are shown in FIGURE 4.1.

FIGURE 4.1 shows that for $D_x/D_y = 1.0$ the amount of negative moment is fairly constant for span ratios greater than 1.0 but when the span ratio is less than 1.0, the amount of negative moment decreases with a drop of about 3.5% of the total moment, Mo, between the span ratio of 1.0 and the span ratio of 0.5. When D_x/D_y is increased above

1.0, the slab is made relatively weaker in the transverse direction. Using the analogy between D_x/D_y and the span ratio this would have an effect similar to increasing the transverse span and decreasing the span ratio. As seen in FIGURE 4.1 for high span ratios, which have little influence on the negative moment, increasing D_x/D_y also has little influence. However, for span ratios less than 1.0, where decreasing the span ratio decreases the negative moment, increasing D_x/D_y also decreases the moment with a decrease of about 3% of Mo between D_x/D_y = 1.0 and D_x/D_y = 4.0. When D_x/D_y is less than 1.0 there is a similar but opposite effect.

The results for the exterior and corner panels shown in FIGURES 4.2, 4.3 and 4.4 are similar to those for the interior panel. The positive moments in a corner panel and perpendicular to the free edge in an exterior panel are constant and are not influenced by either the span ratio or D_x/D_y. The internal and external negative moments vary more than the negative moments for an interior panel with variation of as much as 10% of Mo for the span ratio and 6% of Mo variation between D_x/D_y = 1.0 and D_x/D_y = 4.0. The exterior negative moment increases rather than decreases with a decrease in the span ratio.

4.4.3 Column-Middle Strip Moment Split

FIGURES 4.5 to 4.12 show the column strip moments for negative and positive section for the three types of panels, interior, corner and exterior, for variations in D_x/D_y with $D_x/H = \sqrt{D_x/D_y}$. In each figure the design value for a solid isotropic slab given by proposed

ACI 318-71[5] is shown.

FIGURES 4.5 and 4.6 show that for an interior panel when D_x/D_y = 1.0, the amount of moment in the column strip decreases with an increase in the span ratio. This can be explained by the fact that as the span ratio increases the longitudinal curvature increases relative to the transverse curvature and the transverse curvature becomes unimportant. The curvature in the longitudinal direction becomes fairly uniform across a section and the moment is more uniformly distributed across a section. Therefore, the moment in the column strip will tend towards 50% of the moment in the section as the span ratio tends to infinity.

At the span ratio equal to 1.0 there is a discontinuity in the curve. This is caused by the fact that when the span ratio becomes less than 1.0, the transverse span becomes the long span and since the column strip is one quarter of the short span either side of the column, the width of the column strips now varies with the span ratio. If the column strip had remained one quarter of the transverse span the curve would have continued as a smooth curve for span ratios less than 1.0 with no discontinuity.

When D_x/D_y is increased above 1.0, the slab is stiffened in the longitudinal direction which can be thought of as similar to an increase in the transverse span and a decrease in the span ratio. FIGURES 4.5 and 4.6 show that, as with decreasing the span ratio, increasing D_x/D_y above 1.0 increases the amount of moment in the column strip. Similarly, the amount of moment in the column strip decreases

when D_x/D_y is decreased below 1.0. The increase in column strip moment between $D_x/D_y = 1.0$ and $D_x/D_y = 4.0$ varies from about 8% to 15% of the moment at the section.

Results for corner and exterior panels in FIGURES 4.7 to 4.12 show that the effect of D_x/D_y on the column strip moments in these panels is similar to the effect in the interior panel.

4.5 Variable D_x/H Ratio

4.5.1 Negative-Positive Moment Split

FIGURES 4.13, 4.14 and 4.15 give the amount of moment in the negative section of the slab for variations in the D_x/H ratio for the interior, exterior, and corner panels, respectively.

For a given D_x/D_y ratio the amount of negative moment is not greatly influenced by changes in D_x/H. The maximum variation for the interior panel is only about 1% of the total moment, Mo, for $D_x/D_y = 4.0$. For exterior and corner panels the variation increases slightly with a maximum of about 3.5% of the total moment, Mo, for $D_x/D_y = 4.0$ in the corner panel.

4.5.2 Column-Middle Strip Moment Split

While the effect of variations in the D_x/D_y ratio on the column strip moment was examined, the assumption was made that $D_x/H = \sqrt{D_x/D_y}$ (Huber's approximation). FIGURES 4.16, 4.17 and 4.18 show the effect of D_x/H not equal to $\sqrt{D_x/D_y}$ on moments in the column strip for the three types of panels. The results are shown in terms of a factor, F,

which represents the amount of moment in the column strip at a particular D_x/H value divided by the amount of moment in the column strip when $D_x/H = \sqrt{D_x/D_y}$.

If D_x/H is increased above $\sqrt{D_x/D_y}$, the column strip moment increases, while if D_x/H is decreased below $\sqrt{D_x/D_y}$ the column strip moment decreases. When D_x/H is increased, the twisting stiffness decreases relative to the bending stiffness. As this happens the slab tends to behave more and more like a series of unconnected beams with more of the load being transferred directly to the ends of the beams and into the column strip with less load being carried by the middle strip.

The results for the three types of panels are very similar but in each case there is a fairly wide scatter of values, especially for the positive moments. FIGURE 4.19 is a composite of the results for negative column strip moment for the three types of panel and as can be seen from the figure there is much less scatter. These results can be approximated by the curve shown in the figure.

Since the positive moments are smaller than the negative moments, a large change in the percentage of positive moment in the column strip does not reflect as large a change in the absolute value of the moment as would be the case for negative moment. Therefore, there can be larger errors in the approximation of distribution for positive moments than for negative moments without causing serious problems. Since the scatter for positive moments is on both sides of the negative values, the curve in FIGURE 4.19 can probably be used as a good approximation

for positive as well as negative moments.

4.6 Variable Column Stiffness

In all the previous results the ratio of exterior column to slab stiffness was kept constant, K' = 25. FIGURE 4.20 shows the effect of varying the exterior column stiffness on the negative - positive moment split for moments in a corner panel and for moments perpendicular to the free edge in an exterior panel. The moments in an interior panel and the moments parallel to the free edge for an exterior panel were not effected by variations in the exterior column stiffness.

The external negative moment increases rapidly as the column stiffness increases from zero. At the same time the positive and internal negative moments decrease. However, as the column stiffness becomes larger the changes in moment become smaller and for K' values greater than 10 the positive and negative moments are almost constant. This agrees with the results obtained by Simmonds and Seiss[4] for solid isotropic slabs. The effect of changing D_x/D_y from 1.0 to 4.0 is very small with a maximum variation of about 2% of the total moment, Mo, which can be considered negligible.

4.7 Deflections

The deflections in a slab are effected by the D_x/D, D_y/D, and H/D ratios. However, from the results of the analysis it was observed that the effect of the H/D ratio was small compared to the effect of D_x/D and D_y/D. Therefore, it was thought that it might be possible to approximate the deflections in a cellular slab by using

only the D_x/D and D_y/D ratios. TABLE 4.1 gives the maximum slab deflections for interior, exterior, and corner panels for a typical range of stiffness properties. These deflections are compared to the results obtained by multiplying the deflection for a solid isotropic slab, $D_x/D = D_x/D = H/D = 1.0$, by the average of the inverse of D_x/D and D_y/D, $(D/Dx + D/Dy)/2$.

The agreement for the interior and corner panels is very good. For the interior panel the approximation tends to underestimate the deflections for low values of H/D with the theoretical value a maximum of about 4% higher than the approximate value. For the corner panel the approximation tends to overestimate the deflections with the theoretical deflection about 4% less than the approximate value when $H/D = 1.0$. In the exterior panel the agreement is not so good especially when the value of D_x/D_y is not near 1.0. For $D_x/D_y = 4.0$ the theoretical deflection may be as much as 20% greater than the approximate value.

TABLE 4.1

DEFLECTIONS*

$\frac{D_x}{D}$	$\frac{D_y}{D}$	$\frac{H}{D}$	Magnification Factor (D/Dx + D/Dy)/2	Maximum Deflections					
				Interior Panel		Exterior Panel		Corner Panel	
				Theoretical	Approximate	Theoretical	Approximate	Theoretical	Approximate
1.00	1.00	1.00	1.0	0.058**	0.058	0.082	0.082	0.103	1.103
	1.00	0.20	1.0	0.063	0.058	0.084	0.082	0.104	0.103
	1.00	0.10	1.0	0.064	0.058	0.085	0.082	0.105	0.103
	0.50	1.00	1.5	0.086	0.088	0.135	0.123	0.148	0.154
	0.50	0.20	1.5	0.093	0.088	0.140	0.123	0.154	0.154
	0.50	0.10	1.5	0.095	0.088	0.141	0.123	0.155	0.154
	0.25	1.00	2.5	0.135	0.147	0.237	0.204	0.250	0.257
	0.25	0.20	2.5	0.148	0.147	0.248	0.204	0.253	0.257
	0.25	0.10	2.5	0.151	0.147	0.251	0.204	0.254	0.257
0.50	0.50	0.50	2.0	0.119	0.118	0.162	0.164	0.203	0.205
	0.50	0.10	2.0	0.128	0.118	0.167	0.164	0.206	0.205
	0.50	0.05	2.0	0.129	0.118	0.168	0.164	0.207	0.205
	0.25	0.50	3.0	0.172	0.177	0.268	0.245	0.301	0.308
	0.25	0.10	3.0	0.187	0.177	0.279	0.245	0.306	0.308
	0.25	0.05	3.0	0.190	0.177	0.280	0.245	0.307	0.308
	0.125	0.50	5.0	0.270	0.294	0.472	0.409	0.497	0.513
	0.125	0.10	5.0	0.296	0.294	0.495	0.409	0.504	0.513
	0.125	0.05	5.0	0.302	0.294	0.500	0.409	0.506	0.513

* Deflections are for a span ratio equal to 1.0

** Deflections are given as dimensionless quantities and must be multiplied by the factor, Q. Span ^{4}/10D where Q is the load, to obtain dimensioned values.

TABLE 4.1 (continued)

DEFLECTIONS

$\frac{D_x}{D}$	$\frac{D_y}{D}$	$\frac{H}{D}$	Magnification Factor $(D/D_x + D/D_y)/2$	Maximum Deflections					
				Interior Panel		Exterior Panel		Corner Panel	
				Theoretical	Approximate	Theoretical	Approximate	Theoretical	Approximate
0.25	0.25	0.25	4.0	0.238	0.236	0.323	0.327	0.403	0.411
	0.25	0.05	4.0	0.256	0.236	0.333	0.327	0.409	0.411
	0.25	0.025	4.0	0.259	0.236	0.334	0.327	0.410	0.411
	0.125	0.25	6.0	0.344	0.354	0.535	0.491	0.600	0.616
	0.125	0.05	6.0	0.375	0.354	0.556	0.491	0.610	0.616
	0.125	0.025	6.0	0.381	0.354	0.559	0.491	0.612	0.616
	0.063	0.25	10.0	0.541	0.590	0.943	0.818	0.992	1.027
	0.063	0.05	10.0	0.593	0.590	0.989	0.818	1.005	1.027
	0.063	0.025	10.0	0.605	0.590	0.998	0.818	1.009	1.027
0.10	0.10	0.10	10.0	0.596	0.590	0.807	0.818	1.004	1.027
	0.10	0.02	10.0	0.641	0.590	0.830	0.818	1.019	1.027
	0.10	0.01	10.0	0.650	0.590	0.833	0.818	1.022	1.027
	0.05	0.10	15.0	0.861	0.884	1.333	1.227	1.496	1.540
	0.05	0.02	15.0	0.938	0.884	1.388	1.227	1.520	1.540
	0.05	0.01	15.0	0.955	0.884	1.396	1.227	1.525	1.540
	0.025	0.10	25.0	1.354	1.473	2.357	2.045	2.475	2.570
	0.025	0.02	25.0	1.484	1.473	2.469	2.045	2.508	2.570
	0.025	0.01	25.0	1.514	1.473	2.493	2.095	2.519	2.570

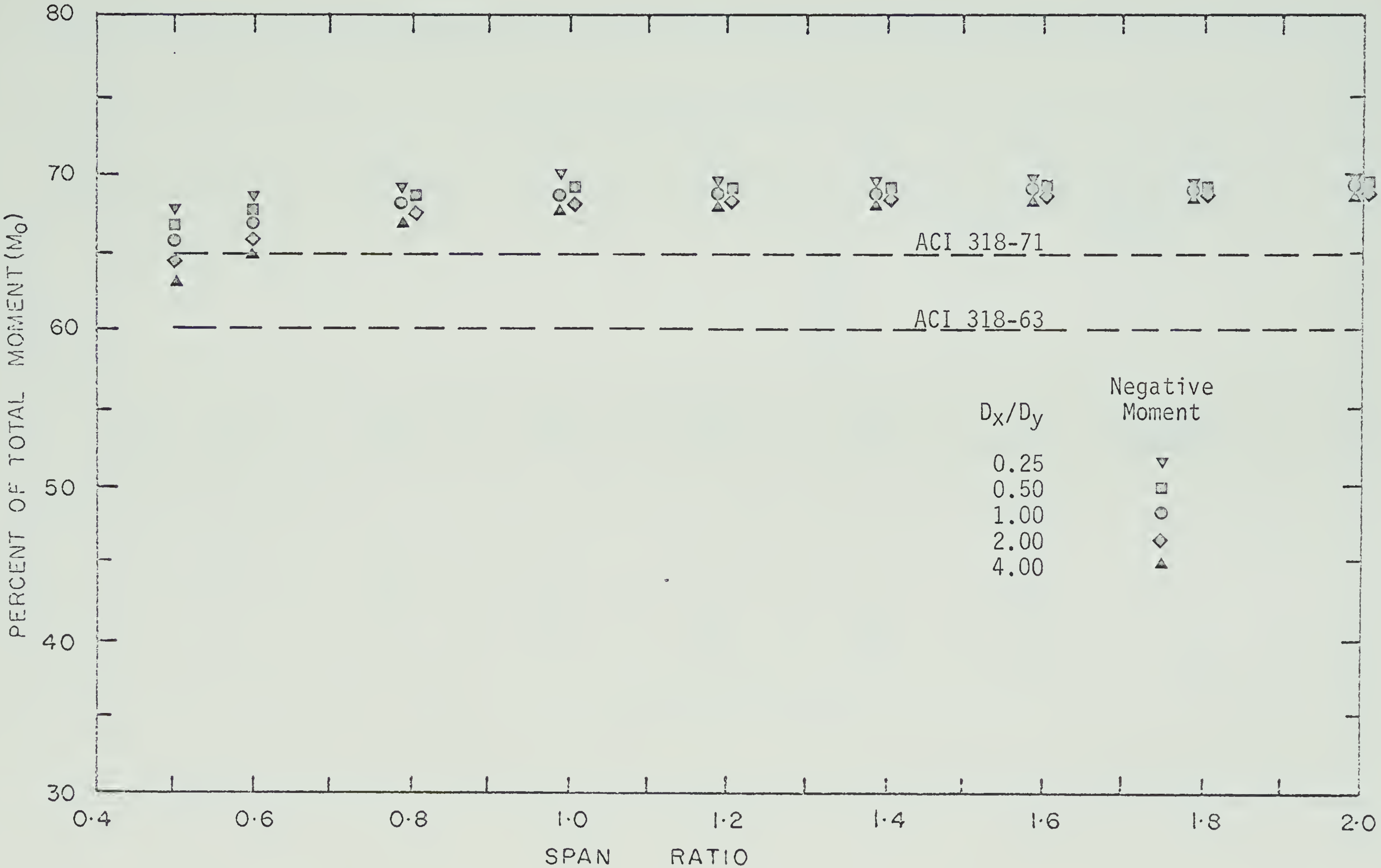


FIGURE 4.1 NEGATIVE-POSITIVE MOMENT SPLIT FOR VARIABLE D_x/D_y RATIO, INTERIOR PANEL

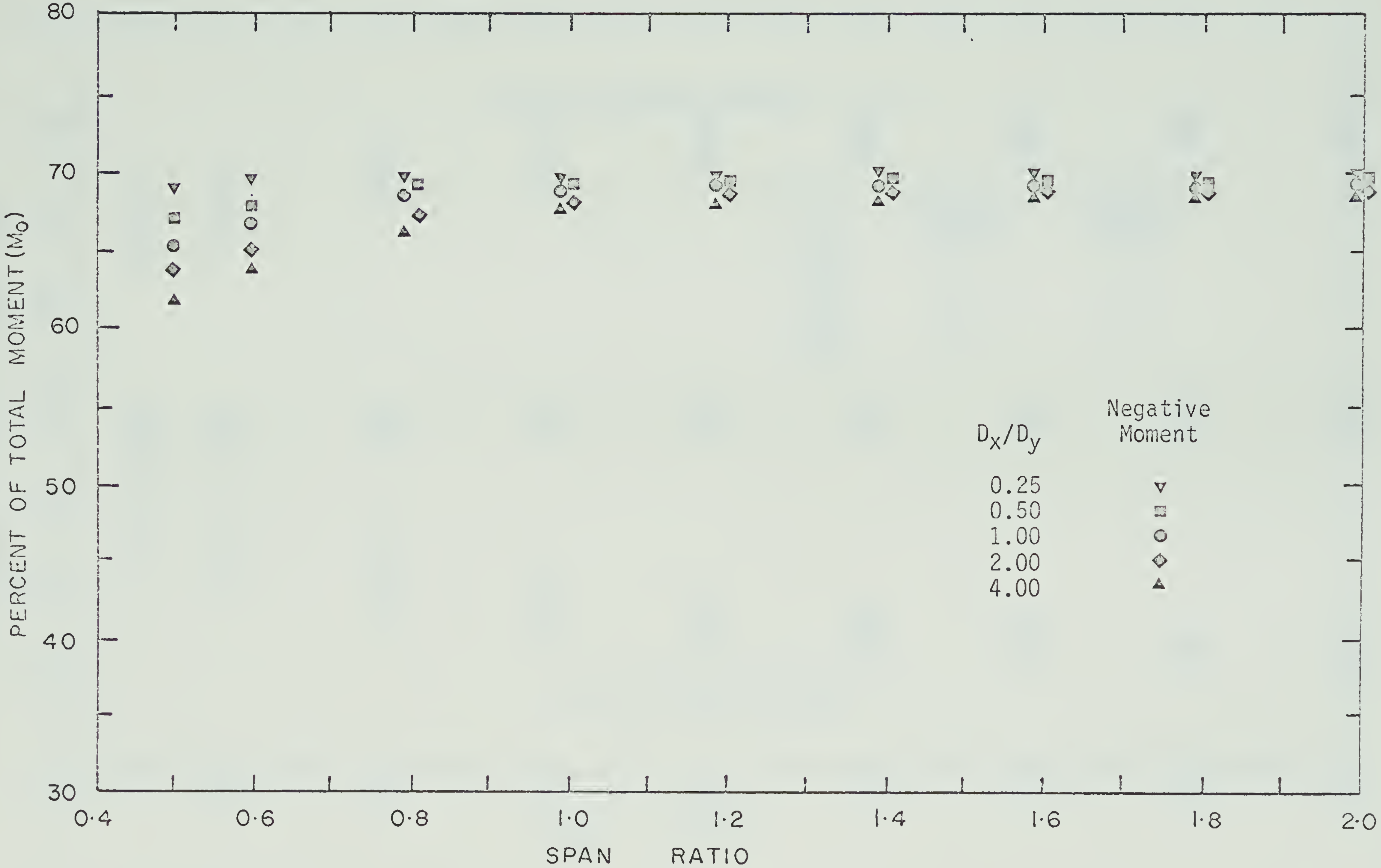


FIGURE 4.2 NEGATIVE-POSITIVE MOMENT SPLIT FOR VARIABLE D_x/D_y RATIO, EXTERIOR PANEL, PARALLEL FREE EDGE

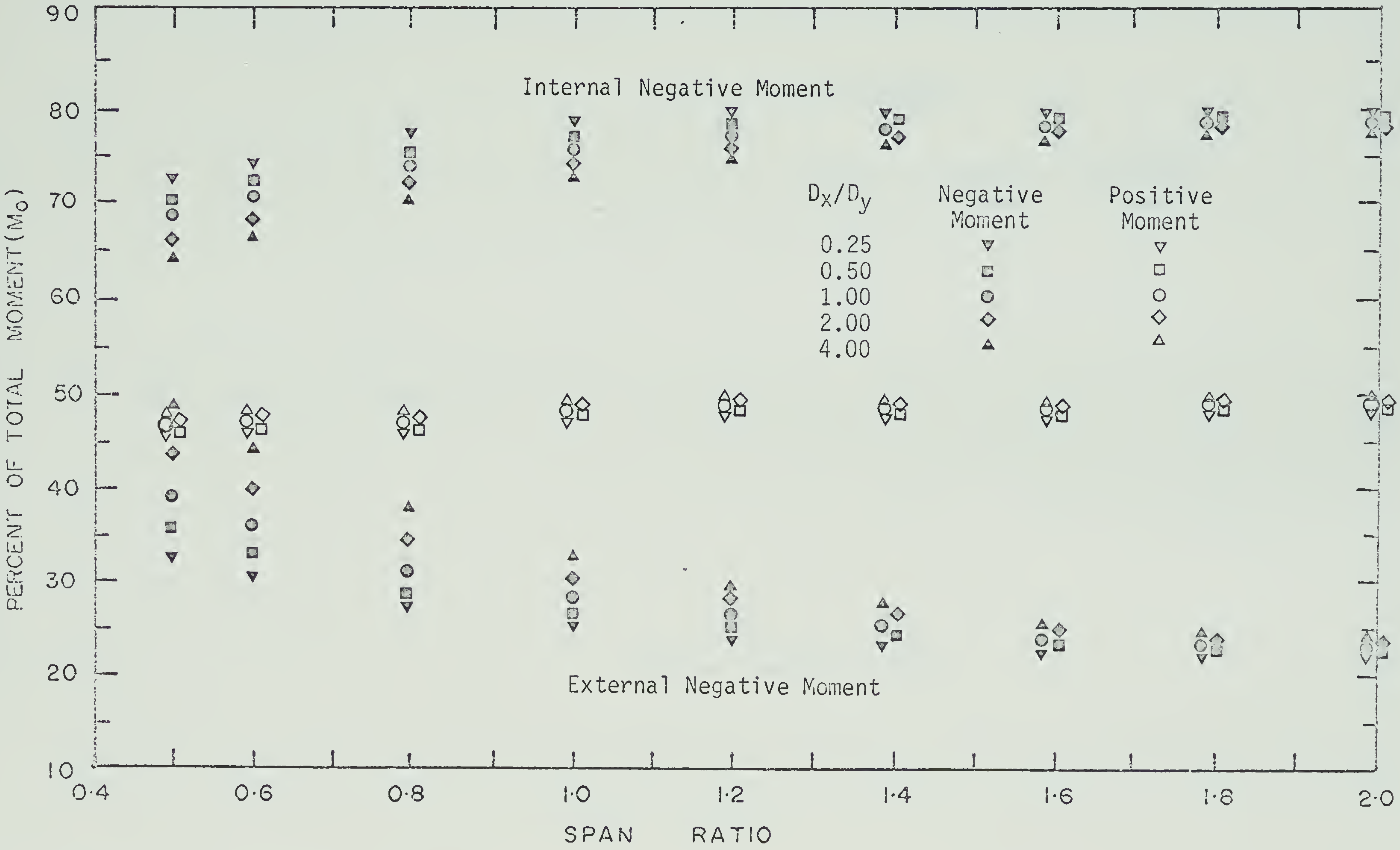


FIGURE 4.3 NEGATIVE-POSITIVE MOMENT SPLIT FOR VARIABLE D_x/D_y RATIO, EXTERIOR PANEL, PERPENDICULAR FREE EDGE

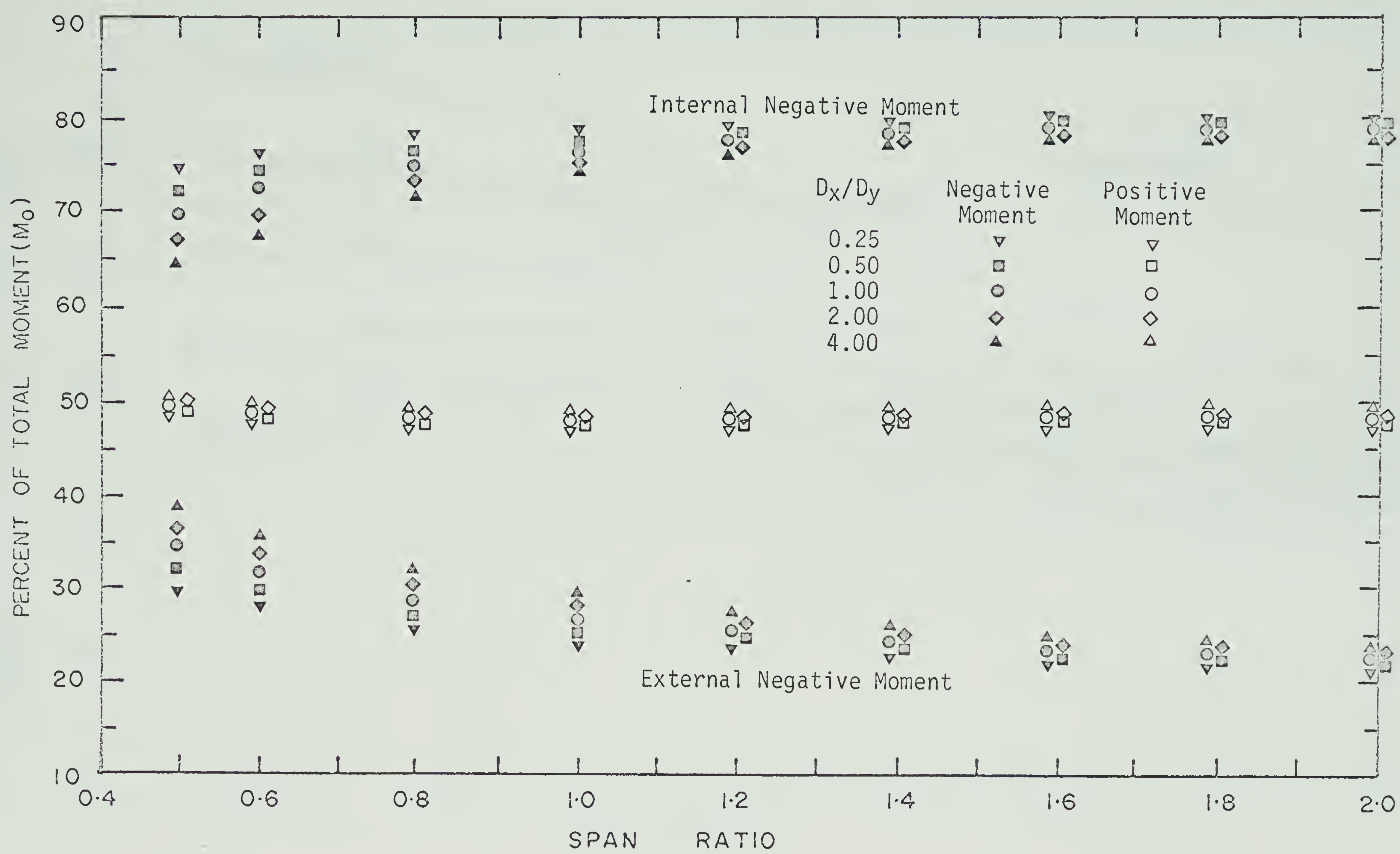


FIGURE 4.4 NEGATIVE-POSITIVE MOMENT SPLIT FOR VARIABLE D_x/D_y RATIO, CORNER PANEL

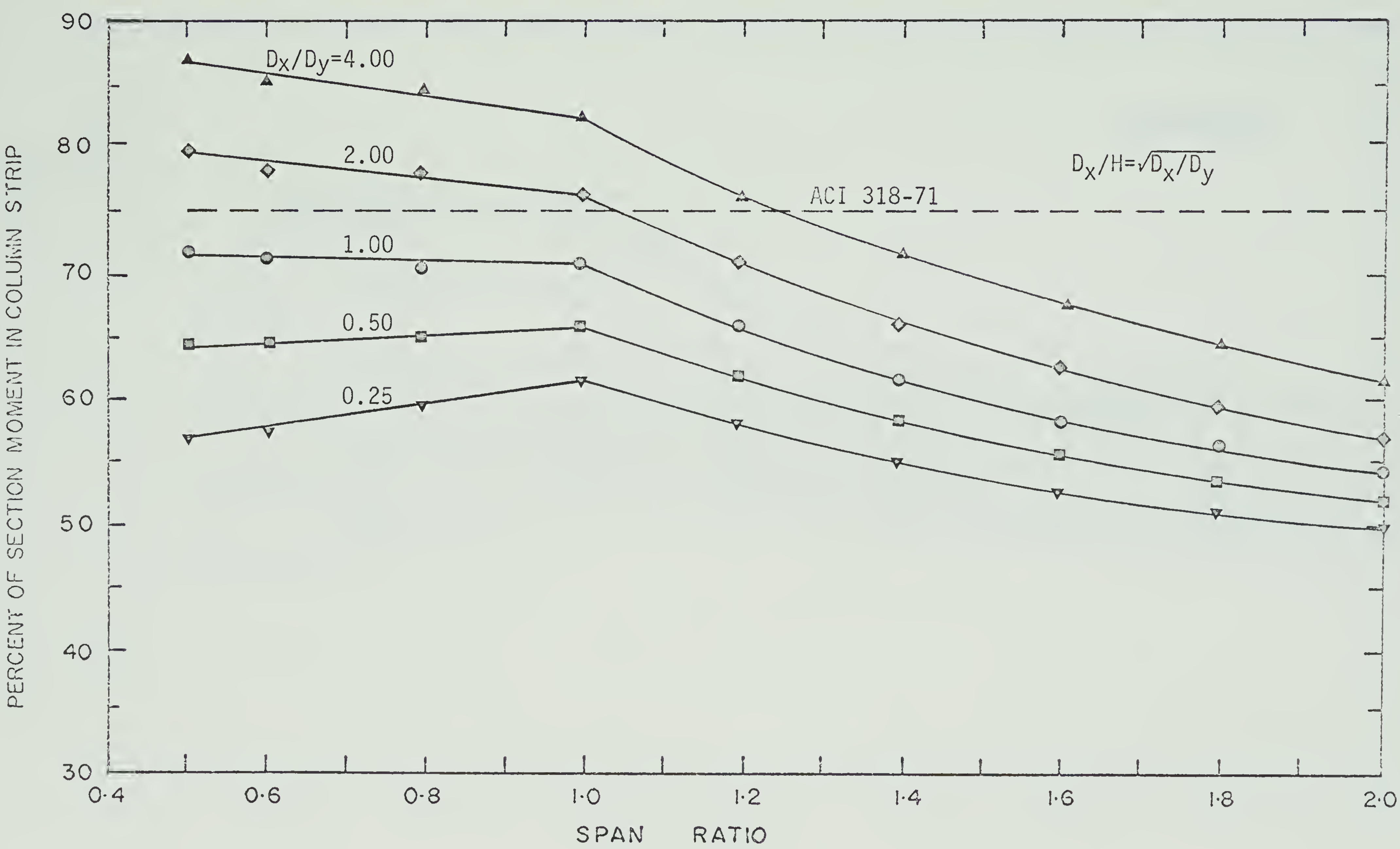


FIGURE 4.5 NEGATIVE COLUMN STRIP MOMENT, INTERIOR PANEL

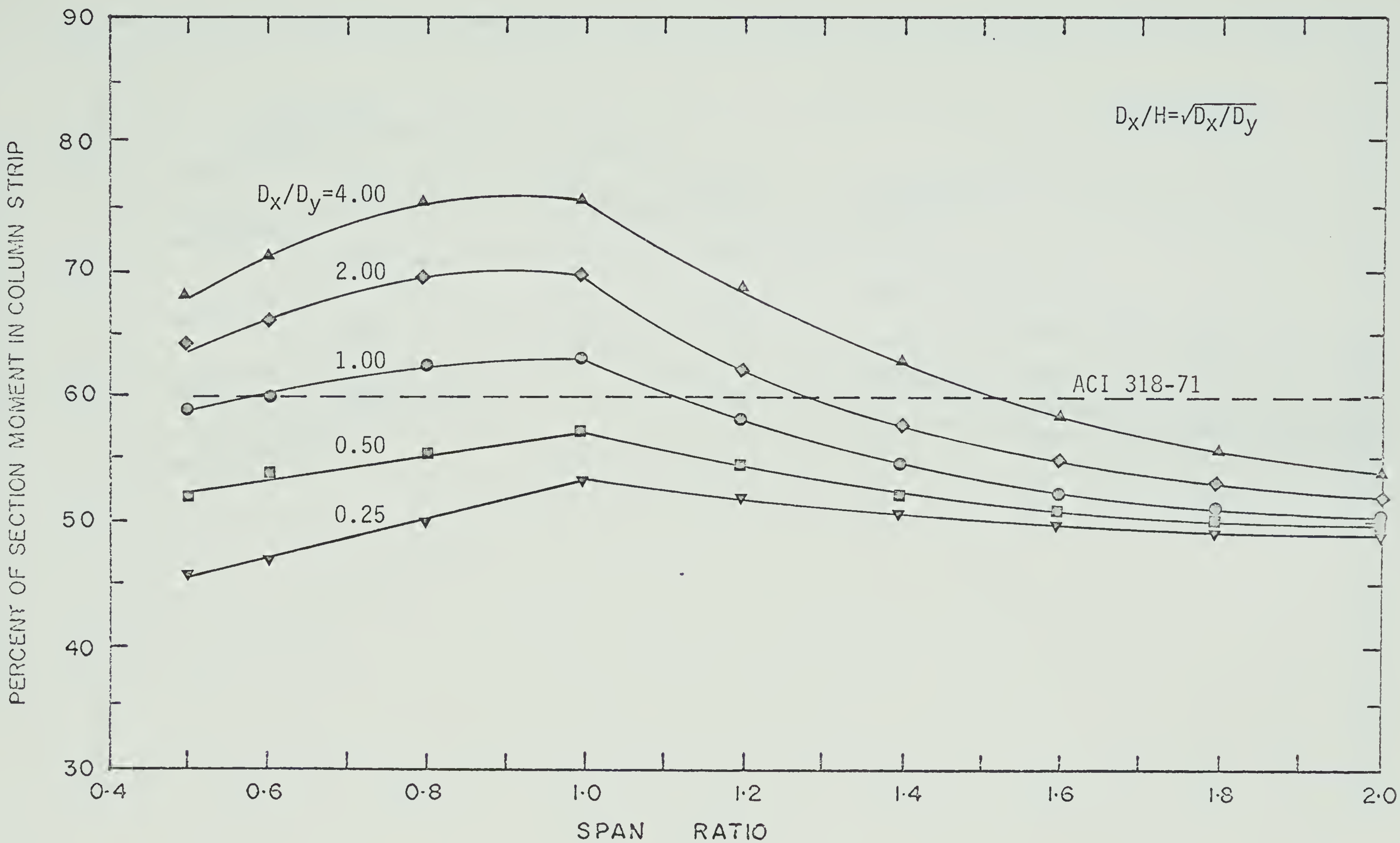


FIGURE 4.6 POSITIVE COLUMN STRIP MOMENT, INTERIOR PANEL

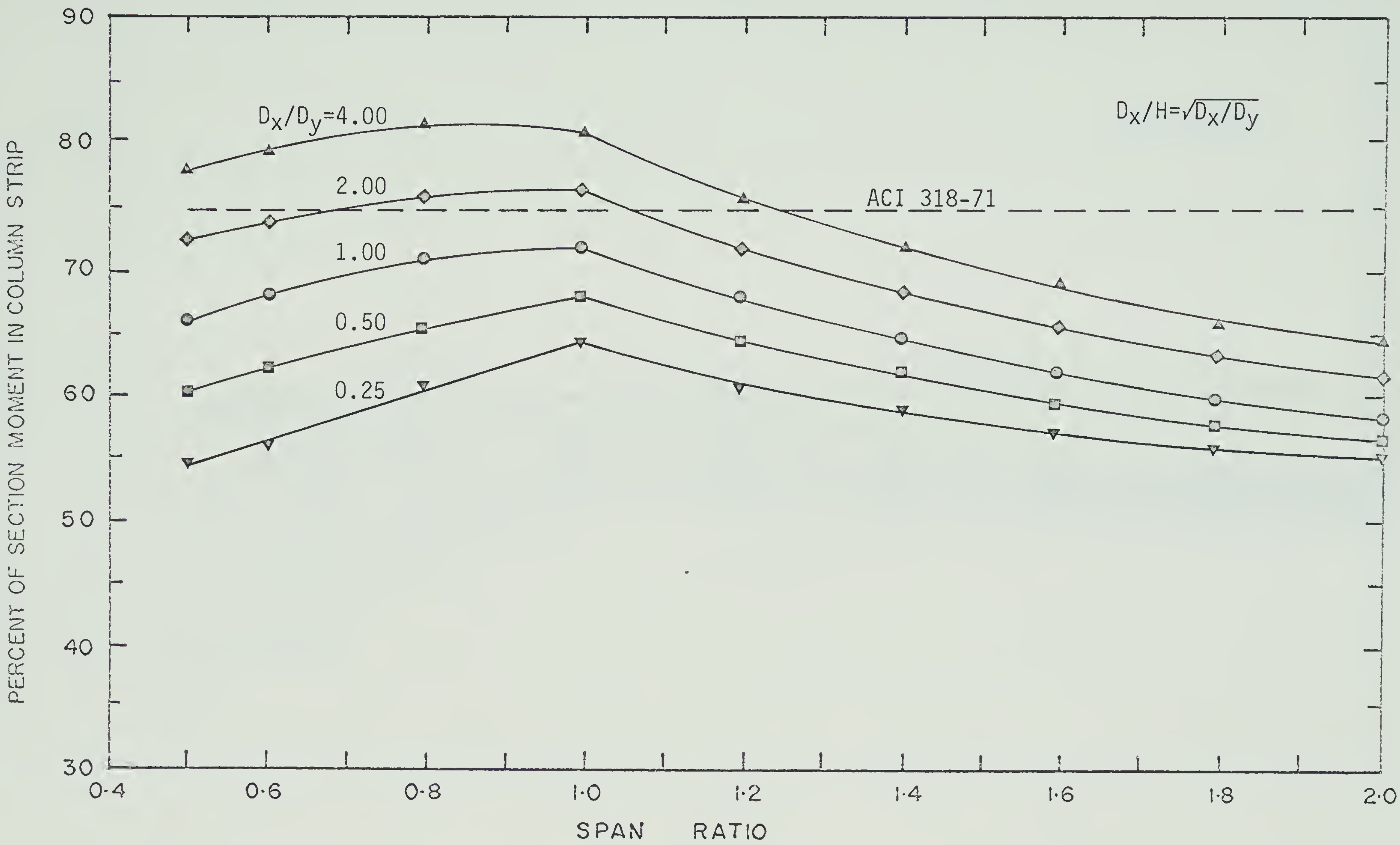


FIGURE 4.7 INTERNAL NEGATIVE COLUMN STRIP MOMENT, CORNER PANEL

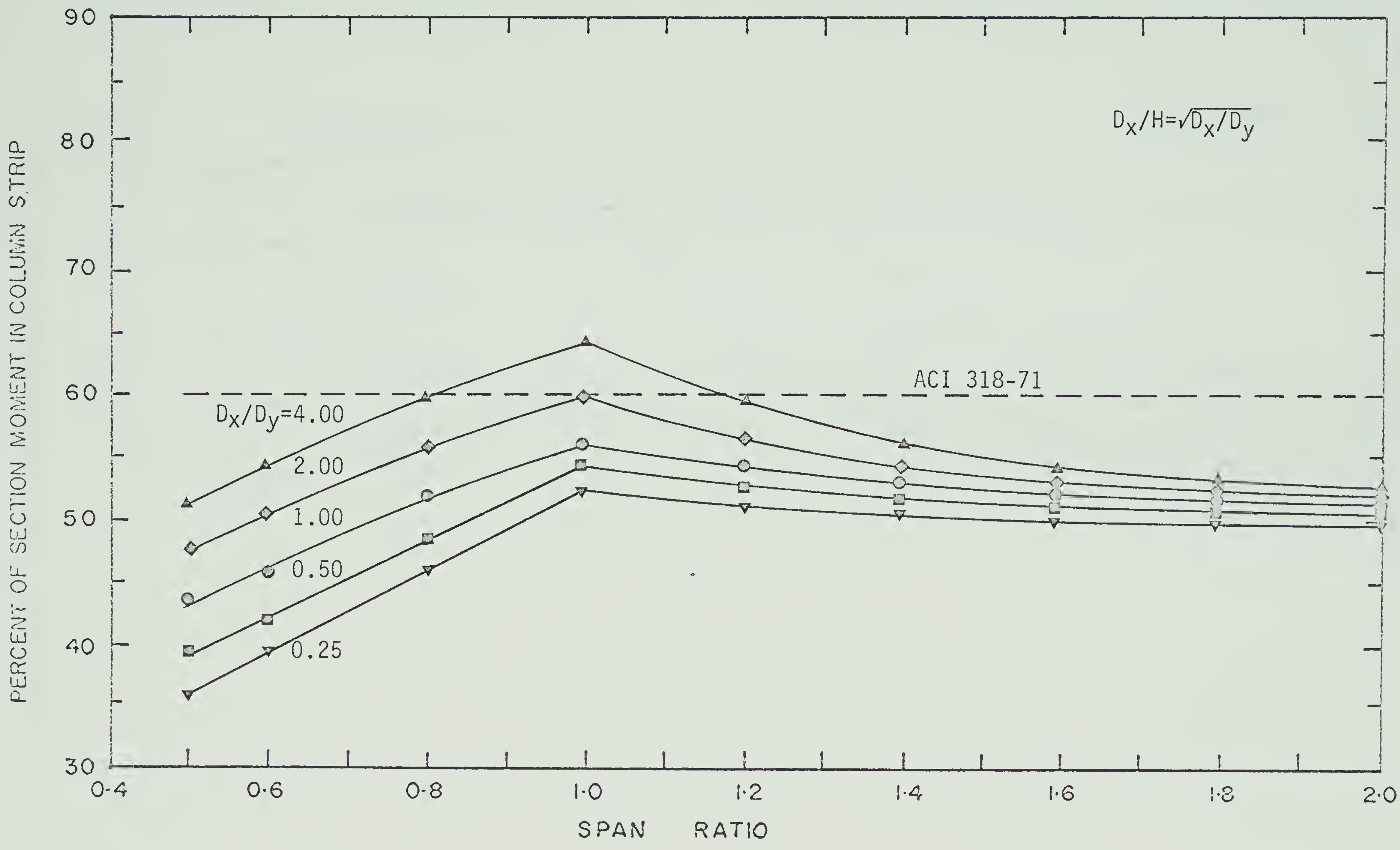


FIGURE 4.8 POSITIVE COLUMN STRIP MOMENT, CORNER PANEL

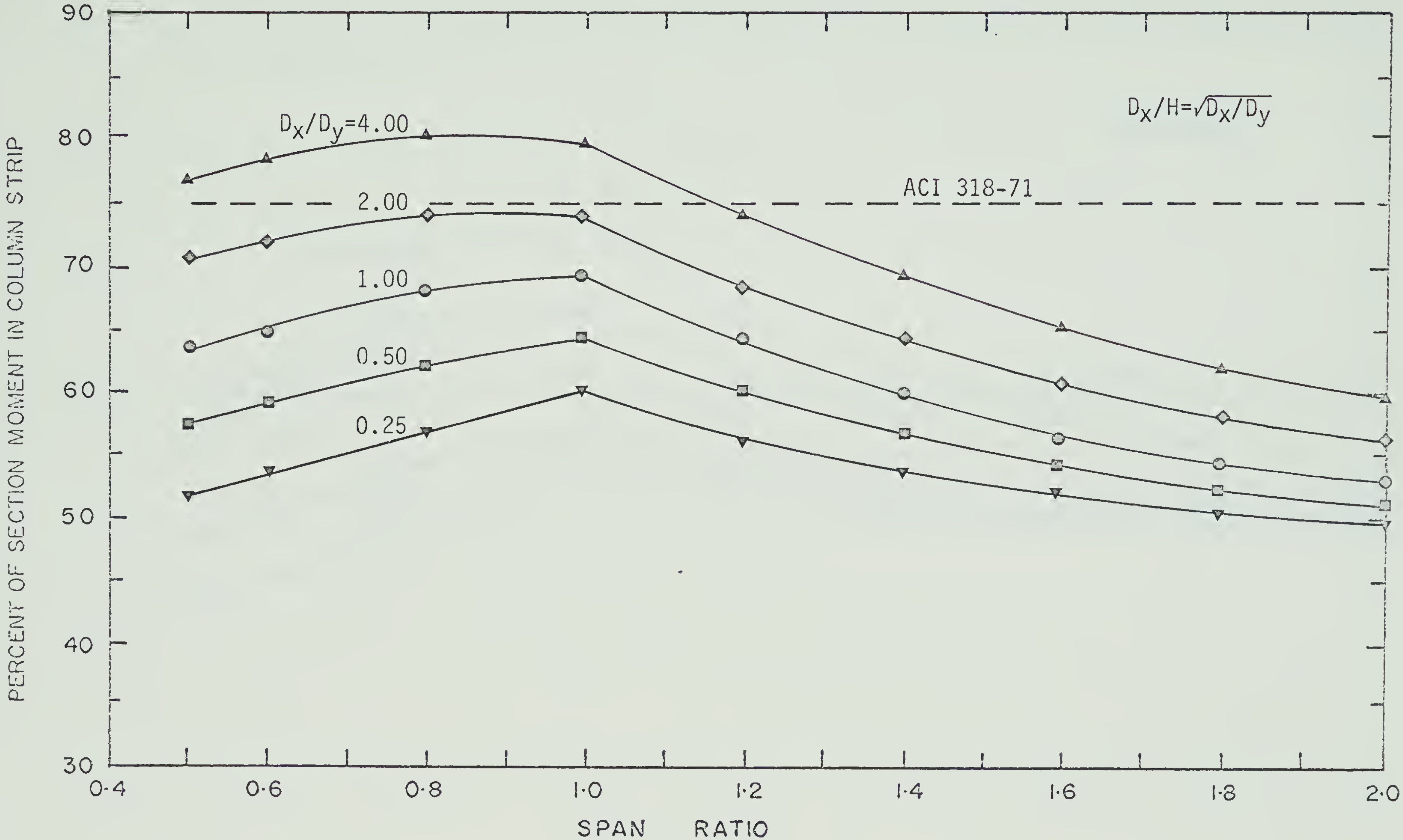


FIGURE 4.9 NEGATIVE COLUMN STRIP MOMENT PARALLEL TO FREE EDGE,EXTERIOR PANEL

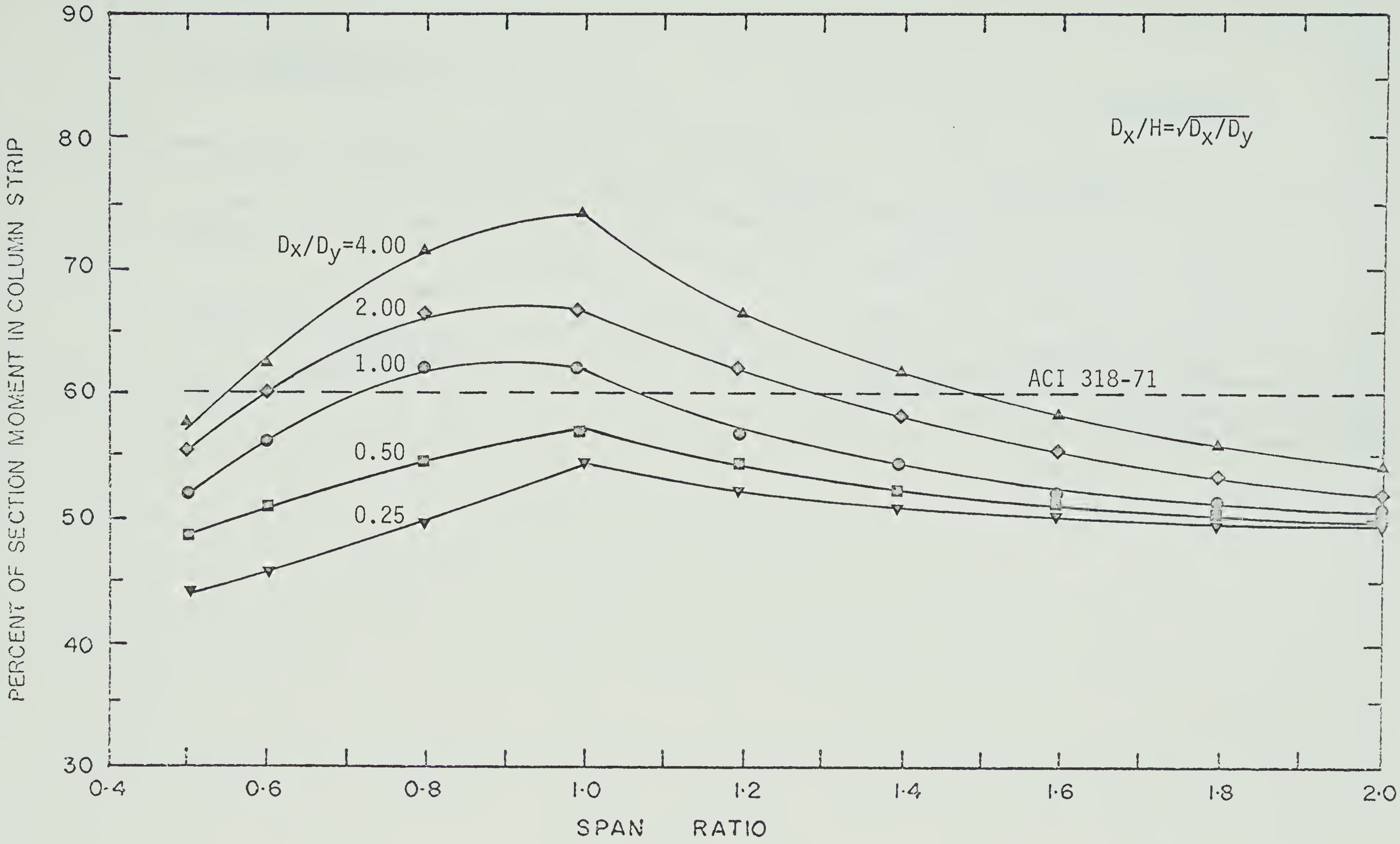


FIGURE 4.10 POSITIVE COLUMN STRIP MOMENT PARALLEL TO FREE EDGE, EXTERIOR PANEL

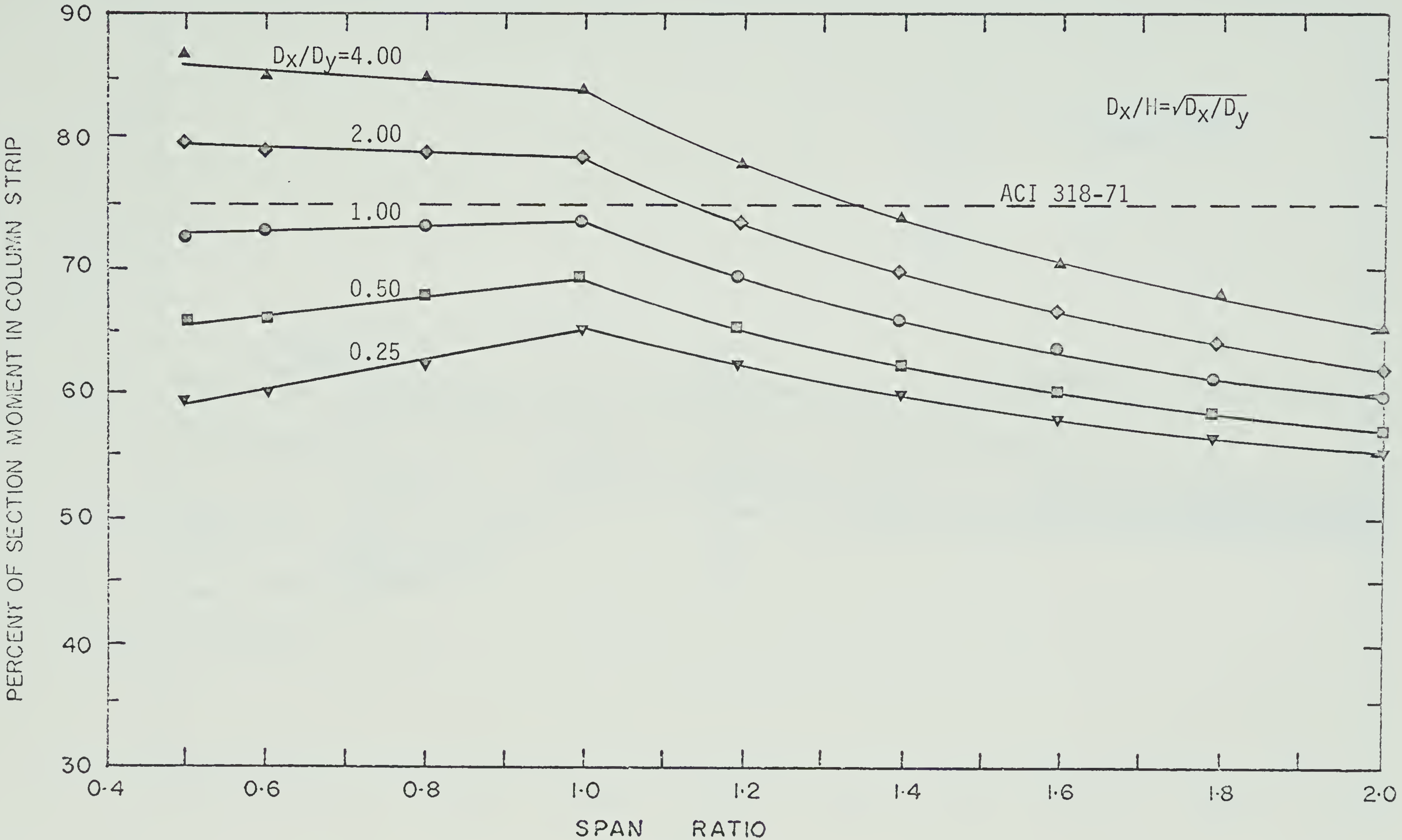


FIGURE 4.11 INTERNAL NEGATIVE COLUMN STRIP MOMENT PERPENDICULAR TO FREE EDGE,EXTERIOR PANEL

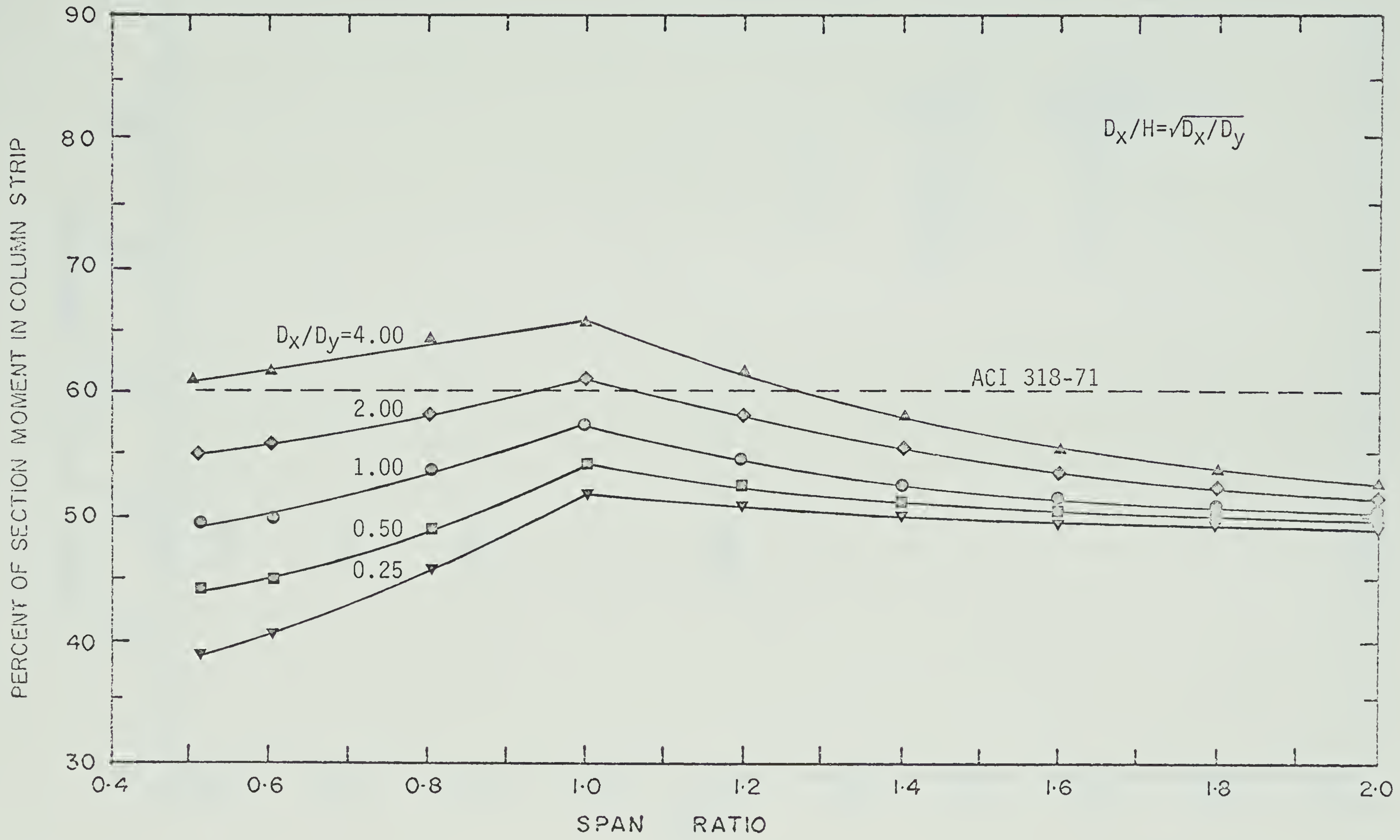


FIGURE 4.12 POSITIVE COLUMN STRIP MOMENT PERPENDICULAR TO FREE EDGE,EXTERIOR PANEL

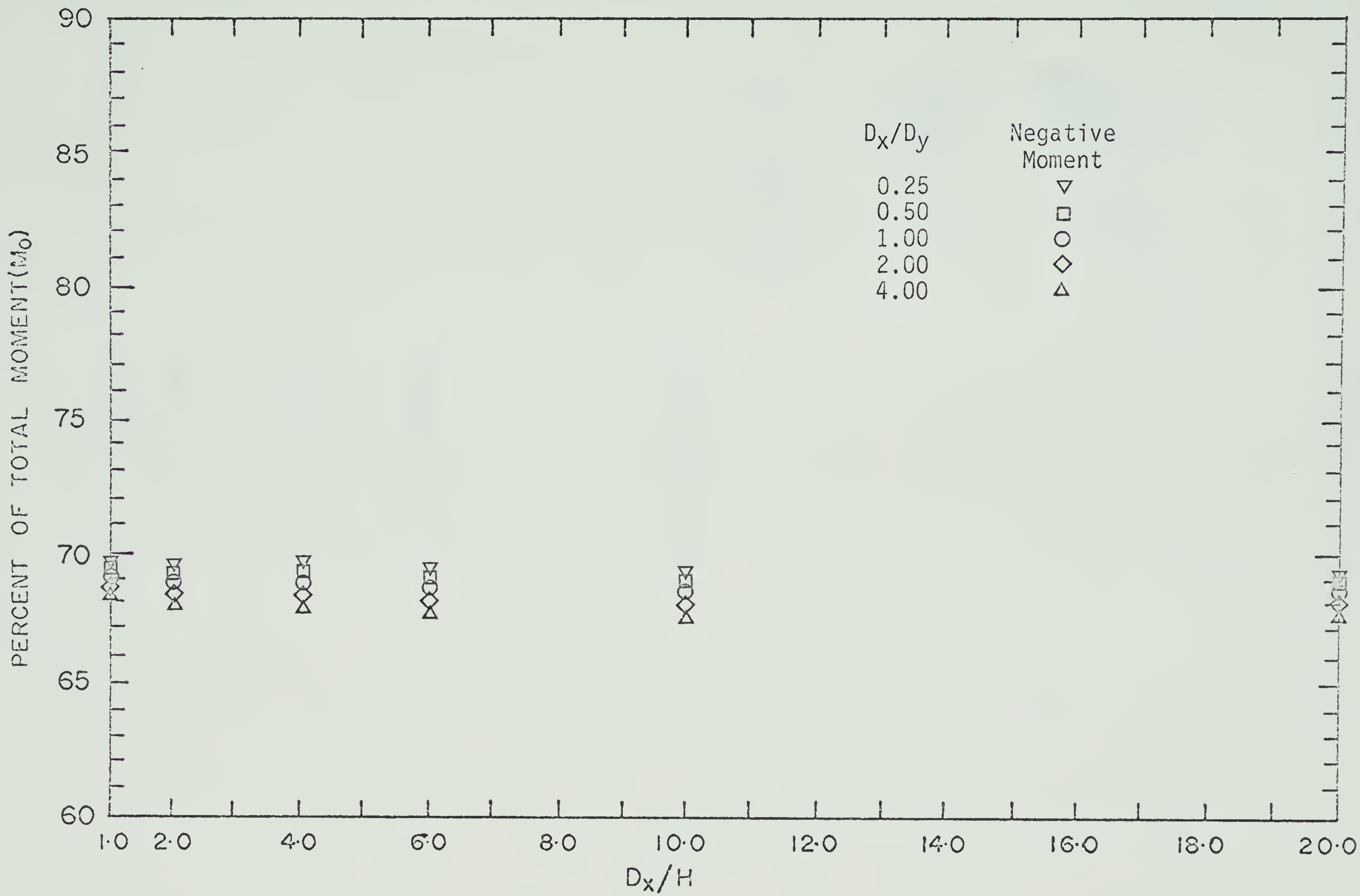


FIGURE 4.13 NEGATIVE MOMENT FOR VARIABLE D_x/H RATIO, INTERIOR PANEL

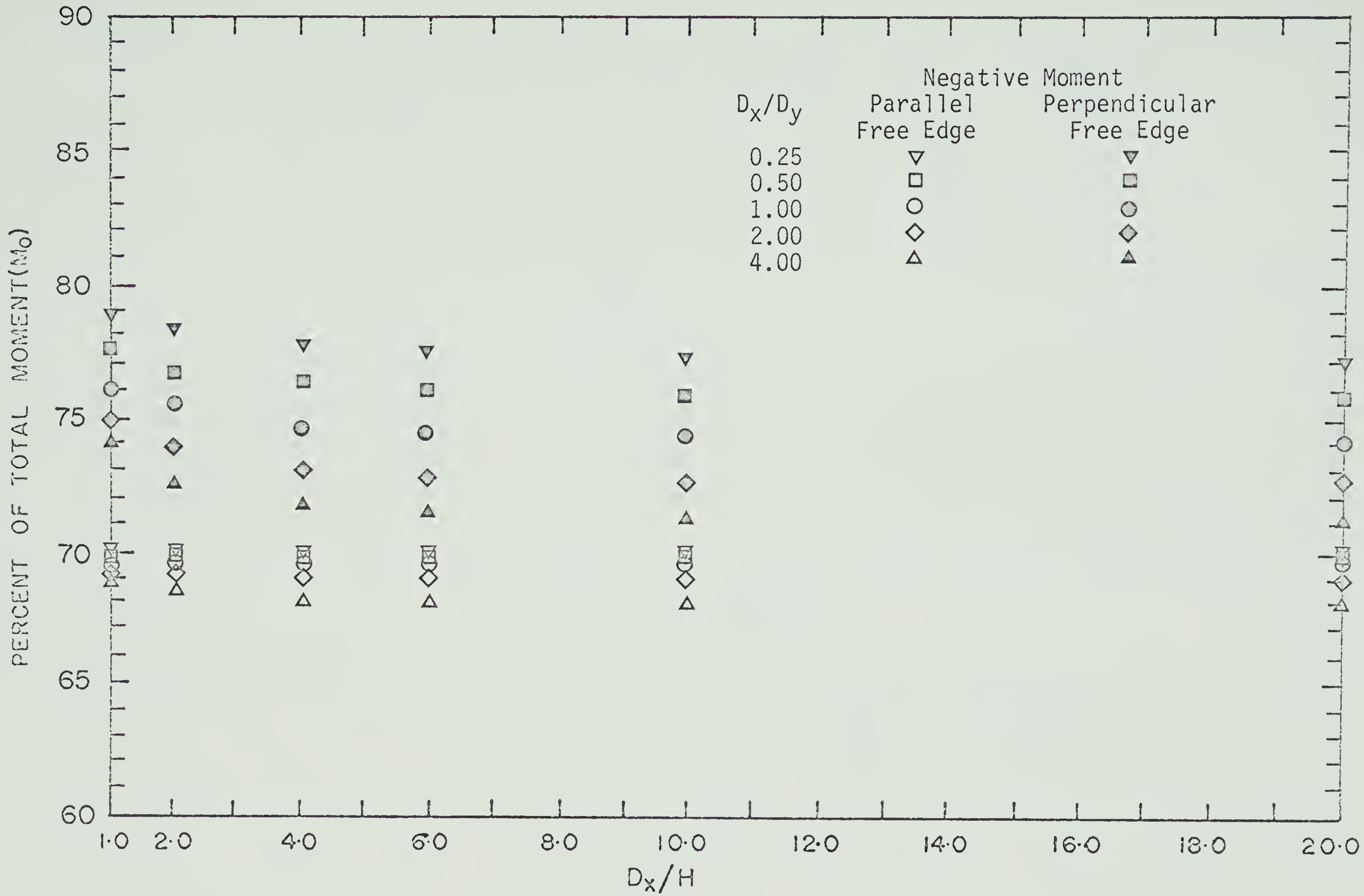


FIGURE 4.14 NEGATIVE MOMENT FOR VARIABLE D_x/H RATIO,EXTERIOR PANEL

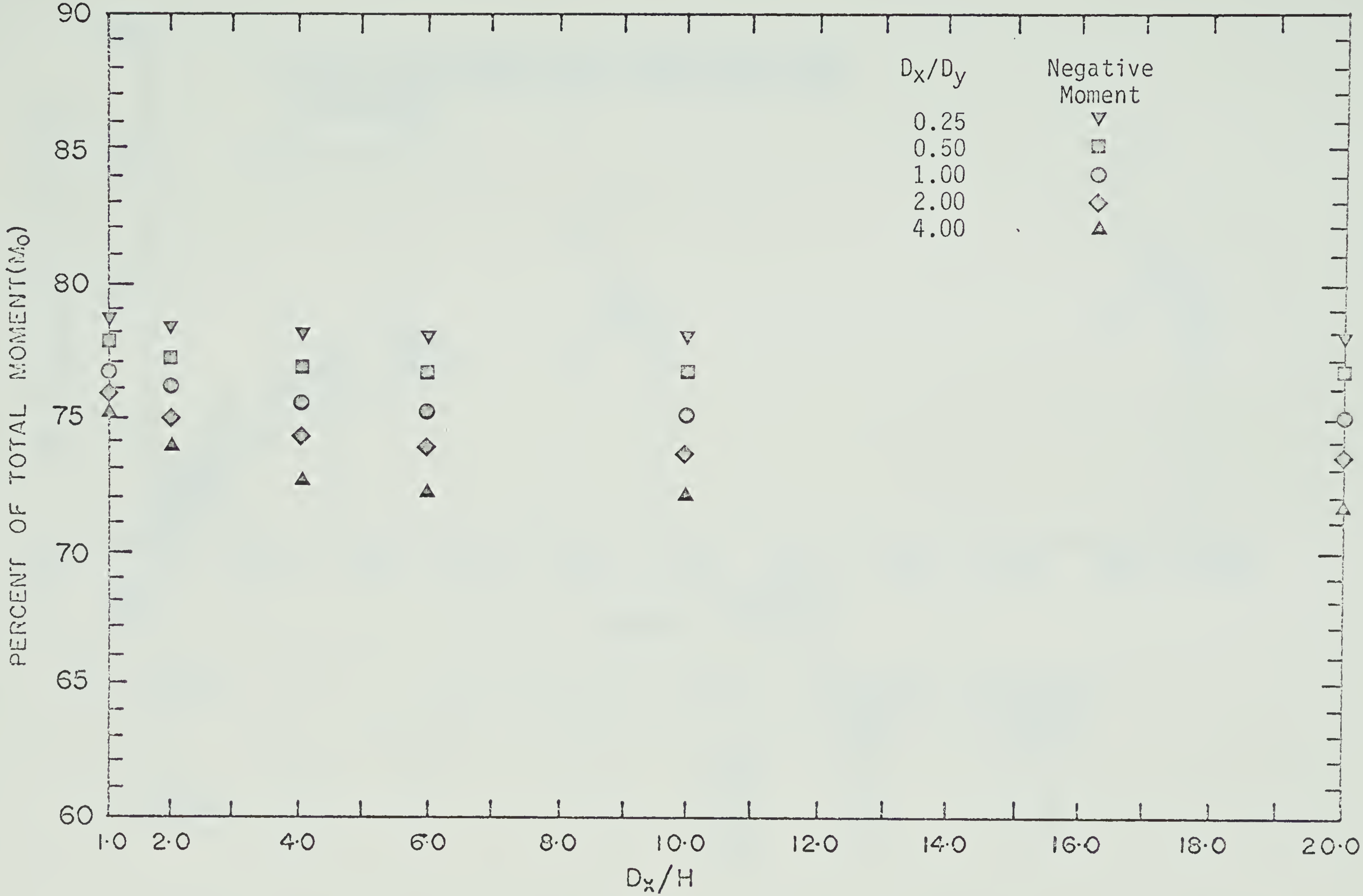


FIGURE 4.15 NEGATIVE MOMENT FOR VARIABLE D_x/H RATIO,CORNER PANEL

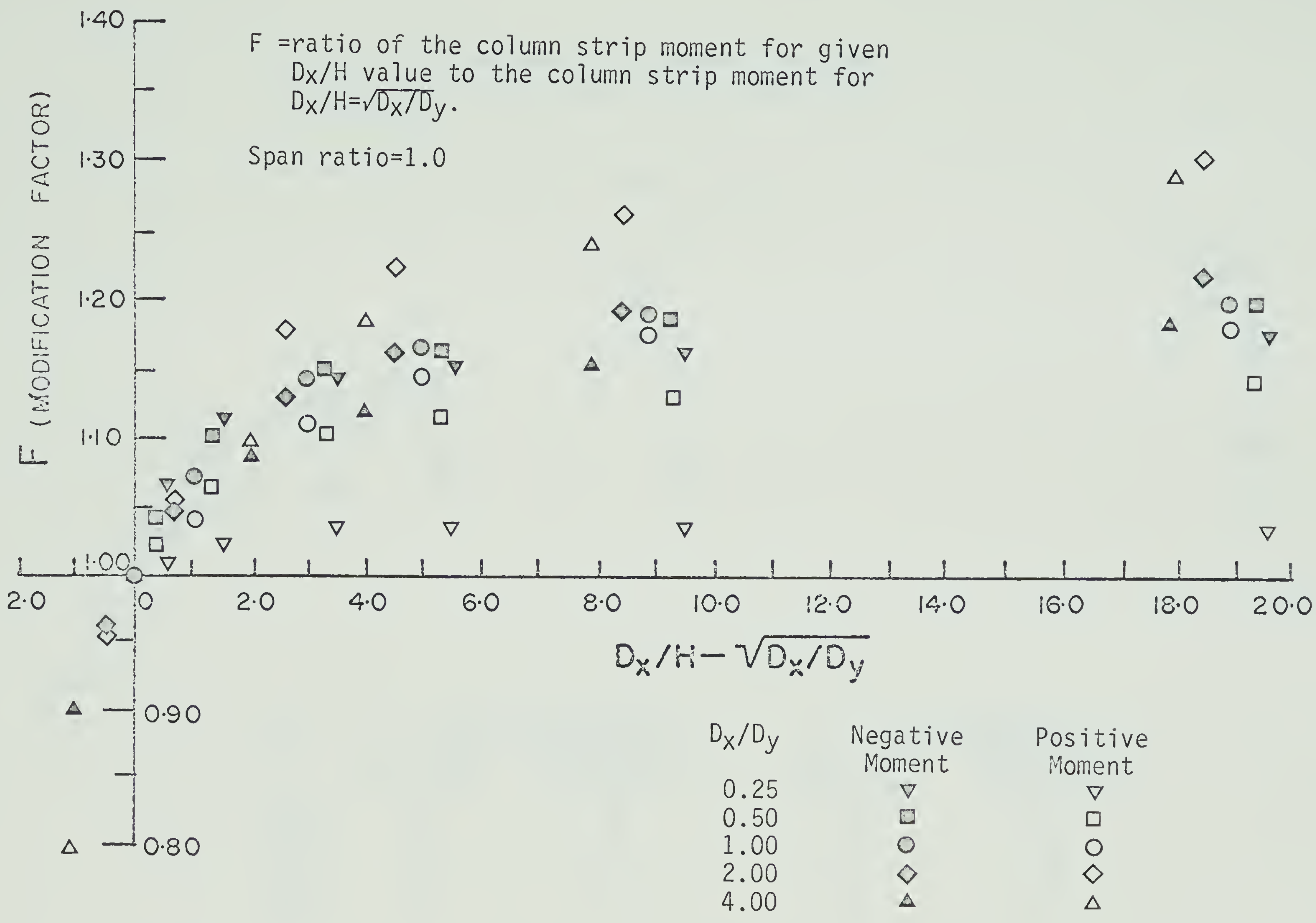


FIGURE 4.16 EFFECT OF VARIATIONS IN D_x/H RATIO ON COLUMN STRIP MOMENT, INTERIOR PANEL

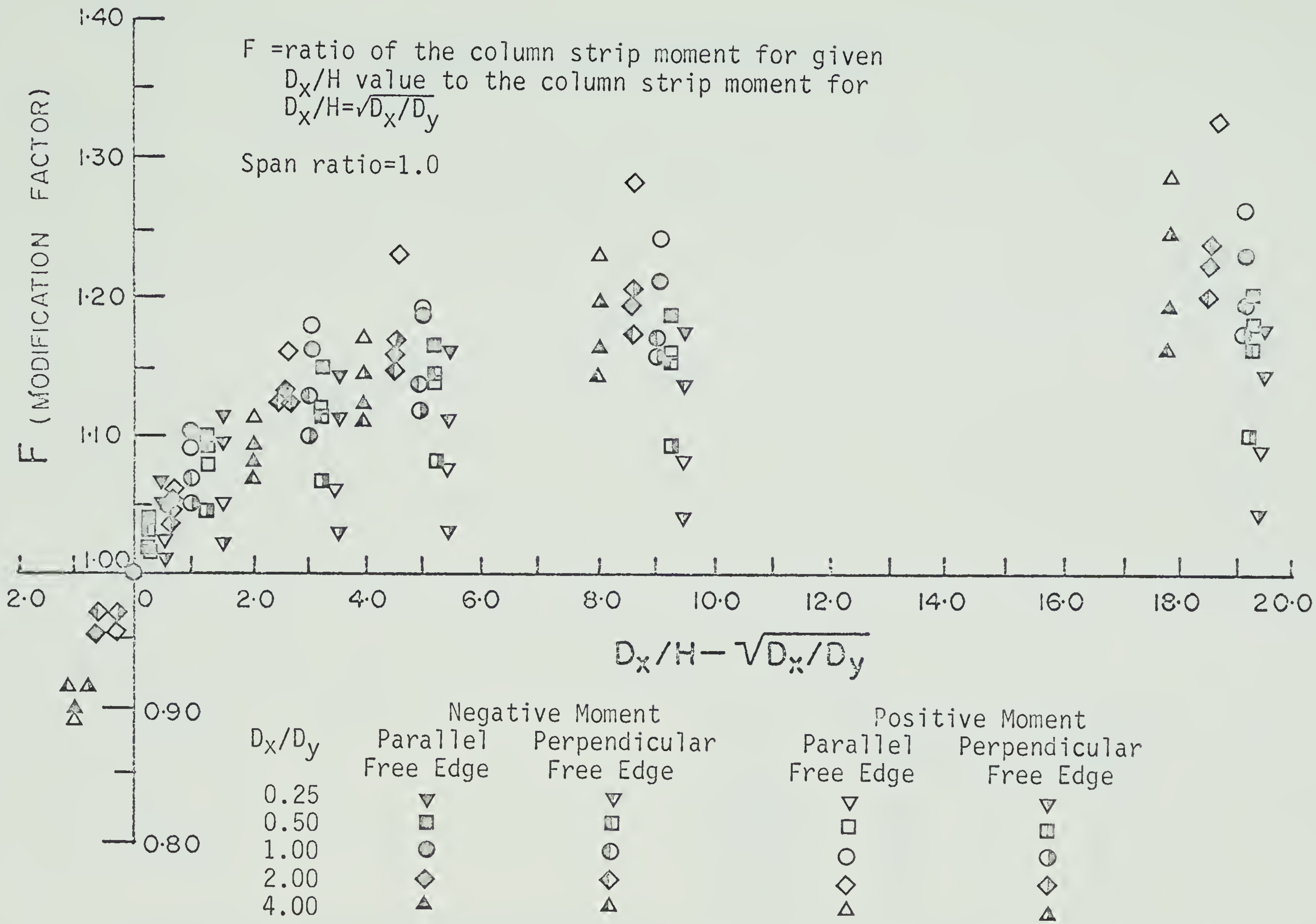


FIGURE 4.17 EFFECT OF VARIATIONS IN D_x/H RATIO ON COLUMN STRIP MOMENT,EXTERIOR PANEL

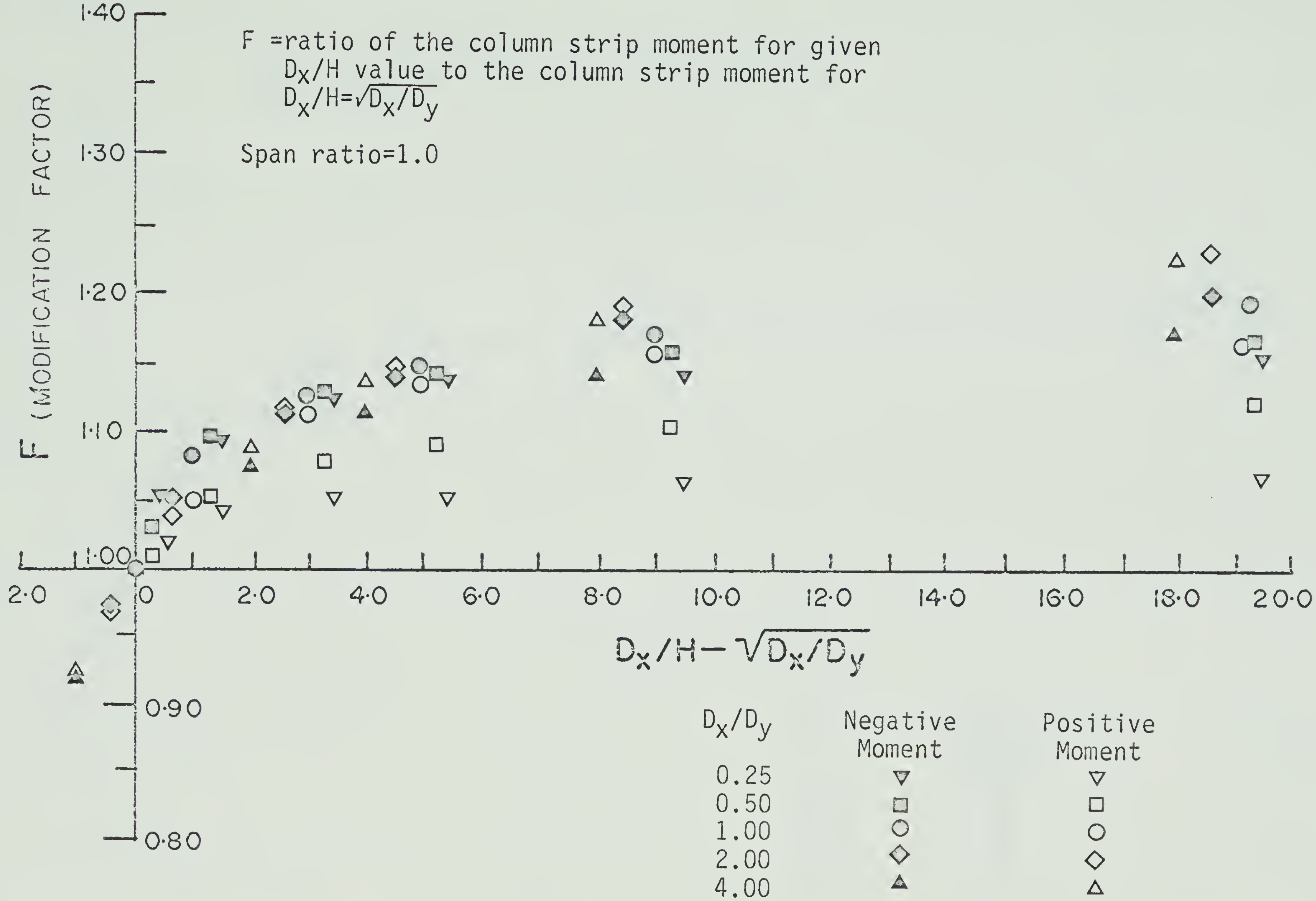


FIGURE 4.18 EFFECT OF VARIATIONS IN D_x/H RATIO ON COLUMN STRIP MOMENT, CORNER PANEL

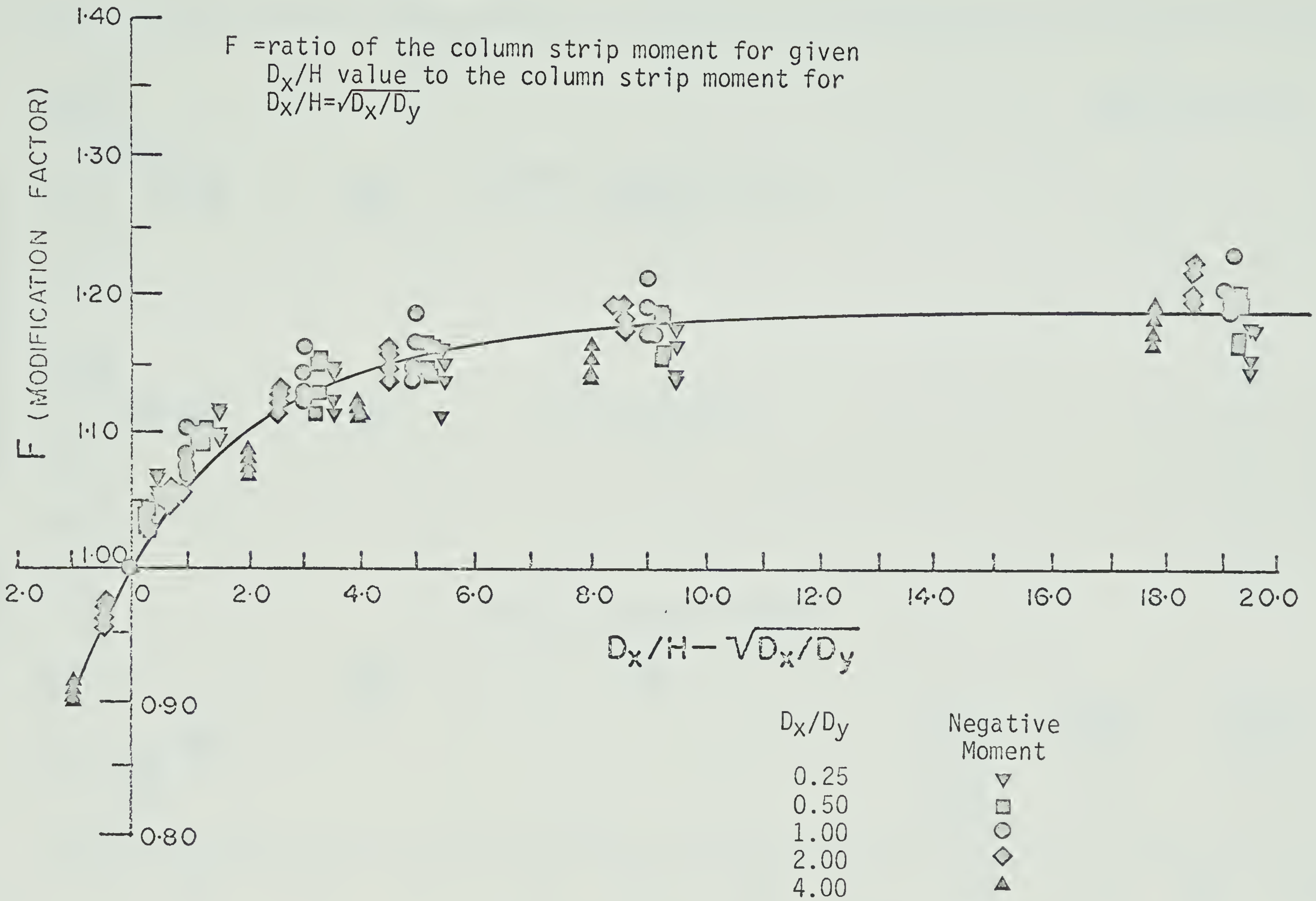


FIGURE 4.19 COLUMN STRIP MOMENT MODIFICATION FACTOR FOR D_x/H RATIO

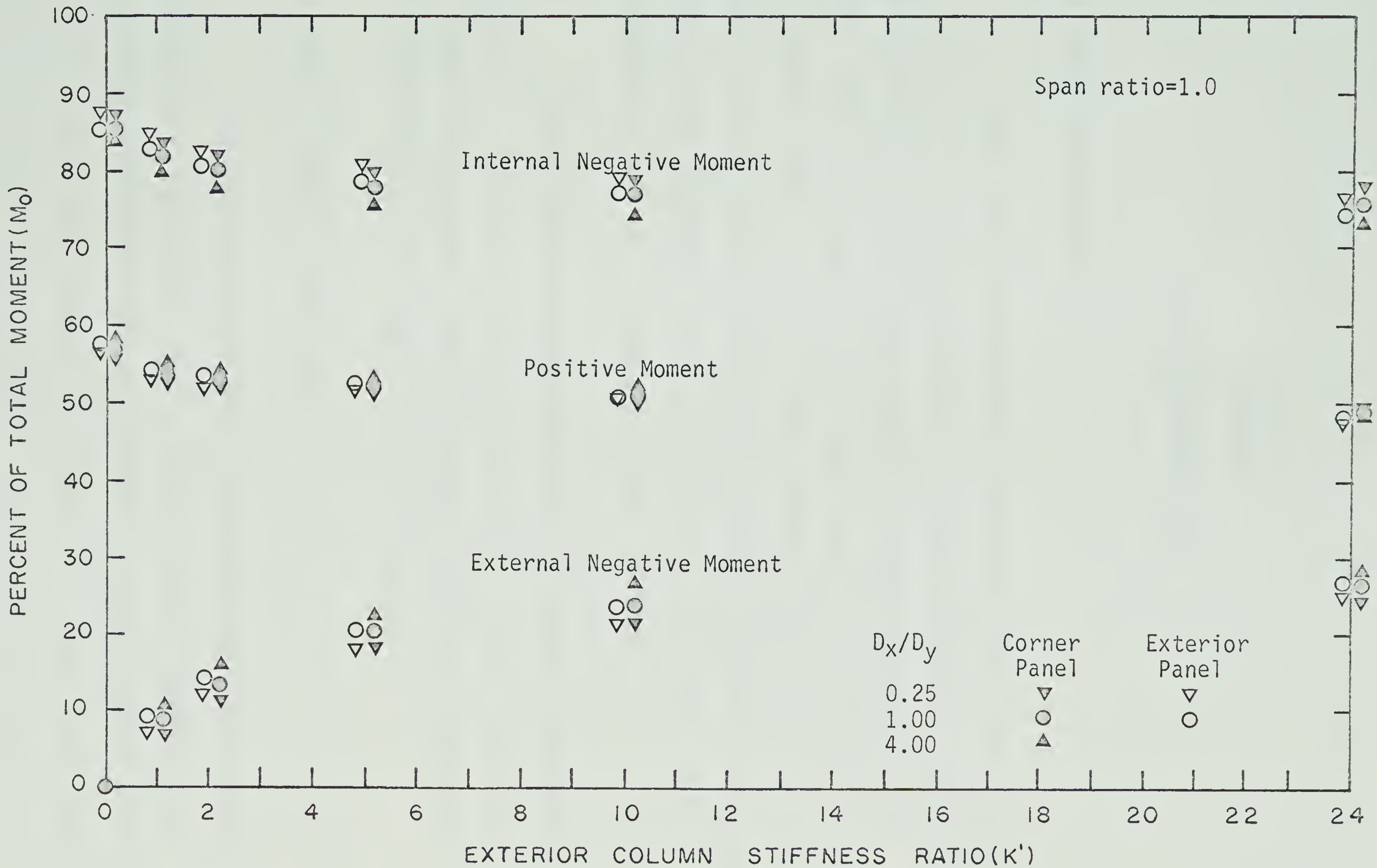


FIGURE 4.20 NEGATIVE AND POSITIVE MOMENT FOR VARIABLE EXTERIOR COLUMN STIFFNESS

CHAPTER V

DESIGN CONSIDERATIONS

5.1 Stiffness Properties

One of the first problems in the design of cellular slabs is to determine the stiffness properties. TABLE 2.1 gives the formulas for the calculation of the stiffness factors for three types of cellular slabs. These three types of slabs would cover most of the cellular slabs that would be built. However, if the stiffness properties for a different type of cellular slab were required, they could be derived using procedures similar to the procedures used for the types of slabs given. These procedures are outlined in CHAPTER II and in APPENDIX B.

The stiffness properties are most conveniently expressed in terms of the dimensionless ratios, D_x/D, D_x/D_y, and D_x/H, where the X-span is always the longitudinal span or the span in which the moments are being determined and the Y-span is the transverse span.

5.2 Negative-Positive Moment Split

Neither the D_x/D_y nor the D_x/H ratio has any great effect on the negative-positive moment split. In all cases the effect of the span ratio on the moment split is as great or greater than the effect of D_x/D_y or D_x/H and in present design practice the span ratio is not considered a factor in determining the negative-positive moment split.

For moments in interior panels and parallel to the free edge in exterior panels the negative-positive split can be assumed constant. For moments in corner panels and perpendicular to the free edge in exterior panels it can be assumed to be a function only of the exterior columns stiffnesses. Therefore, the same criteria use to determine the negative-positive split for solid isotropic slabs can be used for cellular orthotropic slabs.

It should be noted here that the maximum amount of moment that could be carried in the external negative section for exterior and corner panels for very stiff columns was found to be about 30% of the total static moment, Mo. This is considerably less than the value of 65% given in the current draft of the proposed ACI 318-71 Code[5]. The results of this study indicated that it would be impossible to generate 65% of the total moment at the edge of the slab without a torsionally stiff edge beam. Therefore, for a flat plate with no edge beams the proposed ACI 318-71 Code seems to have considerably overestimated the external negative moment.

5.3 Column-Middle Strip Moment Split

The column-middle strip moment split is effected by both the D_x/D_y and the D_x/H ratios. An increase in either D_x/D_y or D_x/H will cause an increase in the column strip moments. The percentage of the moment at a section going to the column strip for D_x/D_y from 0.25 to 4.0, for negative and positive sections, interior, corner and exterior panels, respectively, can be determined from FIGURES 4.5 and 4.12. These values are for a D_x/H ratio equal to $\sqrt{D_x/D_y}$.

When D_x/H is not equal to $\sqrt{D_x/D_y}$, the curve in FIGURE 4.19 can be used to determine a factor, F, which when multiplied by the percentage obtained from FIGURES 4.5 to 4.12, gives values for the particular value of D_x/H.

These values give the percent of the moment at a section carried by the column strip with the remainder being carried by the middle strip.

5.4 Deflections

A good approximation for the deflections for interior and corner panels of a cellular slab can be obtained by multiplying the deflections for a solid isotropic slab of the same thickness by a magnification factor, $(D/D_x + D/D_y)/2$. If the panels are approximately the same size, the corner panel deflections will be the greatest and therefore, the most critical. Since the magnification factor slightly overestimates for a corner panel, it would be excellent to use to check deflections in these cases. In a case where the exterior panel deflection might be critical the deflection obtained by using the magnification factor could be increased by 20% which would cover the maximum variation shown in TABLE 4.1.

5.5 Shear Design

Shear forces were not investigated in this study and no firm conclusions can be made about their effect. However, with all slabs subjected to uniform load the critical section for shear will be around the column and therefore, in cellular slabs it may be necessary to make the slab solid around the columns. This would be similar to using a drop panel or column capital with a solid slab, and if the solid section

is not large compared to the span, it will not greatly effect the distribution of moments in the slab.

5.6 Design Procedure

As a result of this study the following procedure is recommended for determining moments and deflections for the design of cellular orthotropic flat plates with no edge beams on column capitals:

1) Calculate the total static moment, Mo, in the panel as if the slab was isotropic.

2) Split the total moment, Mo, between negative and positive sections using the same method that would be used for a solid isotropic slab.

   Note that the results of this study show that the value for the external negative moment in exterior and corner panels for infinitely stiff exterior columns should be considerably less than the 65% of the total static moment as given by the current draft of the proposed ACI 318-71 Code[5].

3) Calculate the X to Y span ratio and using TABLE 2.1 calculate the stiffness properties, D_x/D, D_x/D_y, and D_x/H. The X-span is always the span for which the moments are being determined.

4) Determine from FIGURES 4.5 to 4.12 the amount of moment to be assigned to the column strip for the D_x/D_y ratio.

5) Modify the amount of moment assigned to the column strip for the D_x/H ratio using the factor, F, from FIGURE 4.19.

6) Repeat the procedure for the transverse span.

7) Check deflections by multiplying the deflections for a solid isotropic slab of the same thickness by the magnification factor, $(D/D_x + D/D_y)/2$.

   Note that as discussed in SECTIONS 4.7 and 5.4 when the exterior panel deflection is critical, this approximation may not give good results and the deflections for this panel may have to be increased by 20%.

5.7 Accuracy of Design Procedure

The preceding design procedure only gives approximate moments which will vary from the actual moments. However, in all cases the procedure designs for 100% of the total static moment. Therefore, if the procedure underdesigns for moment in one area, it will overdesign in another. Since reinforced concrete will allow some redistribution of moment, errors in assigning moments to a particular area of the slab are not critical and the design procedure outlined here should give satisfactory results.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary

The object of this investigation was to study the behaviour of cellular orthotropic slabs and to determine guidelines for their design. The investigation was restricted to cellular flat plates supported on non-deflecting columns with no edge beams or column capitals. The variables considered were the stiffness properties of the slab, the span ratio of the panels, and the exterior columns stiffnesses. The stiffness properties for common types of cellular slabs were determined using a method given by Pfeffer[1].

With the aid of an electronic computer the method of finite differences was used to perform an analysis of a cellular slab.

The behaviour of the slab with variable stiffness properties, span ratio, and exterior columns stiffnesses was studied. Results for a cellular slab were compared to the results for an isotropic slab and an attempt was made to modify current design procedures for isotropic slabs so that they could be used for cellular orthotropic slabs.

6.2 Conclusions

The stiffness properties which distinguish a cellular orthotropic slab from a solid isotropic slab can be expressed in terms of the three ratios, D_x/D, D_x/D_y, and D_x/H. If the slab is isotropic and

solid, the three ratios equal 1.0. The other variables considered in the analysis, the span ratio and the exterior column stiffness, have a similar effect on an orthotropic slab as they have on an isotropic slab.

Within the range of values considered in this investigation, the D_x/D ratio has no effect on the moments in the slab and the only appreciable effect of the D_x/D_y and the D_x/H ratios is to change the distribution of the moments across a section. The total moment at a negative or positive section is not appreciably influenced by the stiffness ratios.

The deflections of a cellular slab are mainly influenced by the bending stiffness ratios, D_x/D and D_y/D. The twisting stiffness also has some effect on slab deflections but it is negligible when compared to the effect of the bending stiffnesses.

For the range of parameters studied in this investigation a cellular orthotropic slab is significantly different from a solid isotropic slab in only two ways, the distribution of moments across a section and the deflections. Therefore, if modifications are made for the column and middle strip moments and for the deflections, a cellular slab can be designed similar to a solid isotropic slab.

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3. Timoshenko, S.P., and Wainowsky-Krieger, S., "*Theory of Plates and Shells*", McGraw-Hill, New York, 1959.

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APPENDIX A

ORTHOTROPIC PLATE EQUATIONS

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ORTHOTROPIC PLATE EQUATIONS

The stress-strain relationships for an orthotropic plate as given by Timoshenko(3) are:

$$\begin{aligned} \sigma_x &= E'_x \varepsilon_x + E'' \varepsilon_y \\ \sigma_y &= E'_y \varepsilon_y + E'' \varepsilon_x \qquad \text{(A1)} \\ \tau_{xy} &= \tau_{yx} = G\gamma_{xy} = G\gamma_{yx} \end{aligned}$$

where the four constants, E'_x, E'_y, E'', and G, characterize the elastic properties of the material. Kirchhoff's assumptions for plates give the following strain-displacement relationships:

$$\varepsilon_x = - z \frac{\partial^2 w}{\partial x^2} \quad , \quad \varepsilon_y = - z \frac{\partial^2 w}{\partial y^2} \quad , \gamma_{xy} = \gamma_{yx} = -2z \frac{\partial^2 w}{\partial x \partial y} \qquad \text{(A2)}$$

By substituting equations (A2) into equations (A1) we obtain the stress-displacement relationships for an orthotropic plate.

$$\begin{aligned} \sigma_x &= -z \left(E'_x \frac{\partial^2 w}{\partial x^2} + E'' \frac{\partial^2 w}{\partial y^2} \right) \\ \sigma_y &= -z \left(E'_y \frac{\partial^2 w}{\partial y^2} + E'' \frac{\partial^2 w}{\partial x^2} \right) \qquad \text{(A3)} \\ \tau_{xy} &= \tau_{yx} = - 2G\, z \frac{\partial^2 w}{\partial x \partial y} \end{aligned}$$

The bending and twisting moments for a plate are given by the following integrals:

$$M_x = \int_{A_x} \sigma_x \; z \; dA_x$$

$$M_y = \int_{A_x} \tau_y \; z \; dA_y$$

$$M_{xy} = \int_{A_x} \tau_{xy} \; z \; dA_x \tag{A4}$$

$$M_{yx} = \int_{A_y} \tau_{yx} \; z \; dA_y$$

Substituting equations (A3) into equations (A4) gives the following expressions:

$$M_x = - E'_x \frac{\partial^2 w}{\partial x^2} \int_{A_x} z^2 dA_x - E'' \frac{\partial^2 w}{\partial y^2} \int_{A_x} z^2 dA_x$$

$$= - \left(E'_x \, I_x \frac{\partial^2 w}{\partial x^2} + E'' \, I_x \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = - E'_y \frac{\partial^2 w}{\partial y^2} \int_{A_y} z^2 dA_y - E'' \frac{\partial^2 w}{\partial x^2} \int_{A_y} z^2 dA_y$$

$$= - \left(E'_y I_y \frac{\partial^2 w}{\partial y^2} + E'' I_y \frac{\partial^2 w}{\partial x^2}\right) \tag{A5}$$

$$M_{xy} = - 2G \frac{\partial^2 w}{\partial x \partial y} \int_{A_x} z^2 dA_x$$

$$= - GJ_x \frac{\partial^2 w}{\partial x \partial y}$$

$$M_{yx} = -2G \frac{\partial^2 w}{\partial x \partial y} \int_{A_y} z^2 dA_y$$

$$= - GJ_y \frac{\partial^2 w}{\partial x \partial y}$$

where I_x and I_y are moments of inertia and J_x and J_y are torsional constants. I_x, I_y, J_x, and J_y are functions of the geometry of the cross-sections of the slab.

From the equations for M_{xy} and M_{yx} in (A5) the torsional constants, J_x and J_y, are as follows:

$$J_x = 2\int_{A_x} z^2 dA_x = 2 I_x$$

$$J_y = 2\int_{A_y} z^2 dA_y = 2 I_y \qquad \text{(A6)}$$

It should be noted that these values for the torsional constants are derived using strain-displacement relationships based on the Kirchhoff's assumptions and are strictly valid only for cellular plates in which the cells are enclosed. For slabs in which the ribs project above or below (see APPENDIX B.3) Kirchhoff's assumptions are not strictly valid and the torsional constants must be modified.

By defining new constants as follows:

$$D_x = E'_x I_x \qquad D_y = E'_y I_y$$

$$D_{1x} = E'' I_x \qquad D_{1y} = E'' I_y \qquad \text{(A7)}$$

$$2D_{xy} = GJ_x \qquad 2D_{yx} = GJ_y$$

and substituting them into equations (A5) we obtain the bending and twisting moment equations for an orthotropic plate. (EQUATION 2.1).

$$
\begin{aligned}
M_x &= -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_{1x} \frac{\partial^2 w}{\partial y^2}\right) \\
M_y &= -\left(D_y \frac{\partial^2 w}{\partial y^2} + D_{1y} \frac{\partial^2 w}{\partial x^2}\right) \\
M_{xy} &= -2D_{xy} \frac{\partial^2 w}{\partial x \partial y} \\
M_{yx} &= -2D_{yx} \frac{\partial^2 w}{\partial x \partial y}
\end{aligned}
\qquad (A8)
$$

APPENDIX B

STIFFNESS CONSTANTS FOR CELLULAR SLABS

APPENDIX B

STIFFNESS CONSTANTS FOR CELLULAR SLABS

B.1 Introduction

As stated in SECTION 2.2 cellular slabs get their orthotropic character from the geometry of their cross-sections. The elastic properties of the material are isotropic. That is:

$$E'_x = E'_y = \frac{E}{1-\nu^2}, \quad E'' = \frac{\nu E}{1-\nu^2}, \quad G = \frac{E}{2(1+\nu)} \qquad \text{(B1)}$$

Therefore, by substituting equations (B1) into equations (A7) and comparing with equations (2.11) we obtain the following relationships:

$$\begin{aligned}
D_x &= \frac{E}{1-\nu^2}\, I_x = \frac{E}{1-\nu^2}\cdot\frac{h^3}{12}\cdot C_x \\
D_y &= \frac{E}{1-\nu^2}\, I_y = \frac{E}{1-\nu^2}\cdot\frac{h^3}{12}\cdot C_y \\
2D_{xy} &= \frac{E}{2(1+\nu)}\cdot J_x = \frac{E}{2(1+\nu)}\cdot\frac{h^3}{12}\cdot 2C_{xy} \\
2D_{yx} &= \frac{E}{2(1+\nu)}\cdot J_y = \frac{E}{2(1+\nu)}\cdot\frac{h^3}{12}\cdot 2C_{yx}
\end{aligned} \qquad \text{(B2)}$$

where I_x and I_y are the bending moment of inertia terms and J_x and J_y are the torsional constants.

From equations (B2) it can be seen that:

$$I_x = \frac{h^3}{12} \cdot C_x \qquad I_y = \frac{h^3}{12} \cdot C_y$$

$$J_x = \frac{h^3}{12} \cdot 2C_{xy} \qquad J_y = \frac{h^3}{12} \cdot 2C_{yx} \tag{B3}$$

Therefore, if the moments of inertia and the torsional constants can be calculated for a cellular slab, the stiffness constants, C_x, C_y, C_{xy}, and C_{yx}, can be evaluated.

B.2 Cylindrical Cell Units

The type of cellular slab being considered here is shown in FIGURE B.1. The characteristics of the cross-sections can be given by expressions similar to the ones used for a prismatic cellular slab.

$$\delta_x = \frac{h_i}{b_x}, \quad \delta_y = \frac{b_{iy}}{b_y}, \quad \lambda = \frac{h_i}{h} \tag{B4}$$

The bending moment of inertia of the cross-section in the X-direction is equal to the moment of inertia of a solid rectangular section minus the moment of inertia of a circular section.

Therefore,

$$I_x = \frac{h^3}{12} C_x = \frac{h^3}{12} - \frac{1}{b_x} \cdot \frac{\Pi h_i{}^4}{64}$$

$$= \frac{h^3}{12} \left(1 - \frac{3\Pi}{16} \cdot \frac{h_i}{b_x} \cdot \left(\frac{h_i}{h}\right)^3\right)$$

$$= \frac{h^3}{12} \left(1 - \frac{3\pi}{16} \delta_x \right)$$

and

$$C_x = \left(1 - \frac{3\pi}{16} \delta_x \right) \tag{B5}$$

When calculating the moment of inertia for prismatic cells, the width of the rib perpendicular to the direction of view was neglected. This is no longer a good approximation when calculating the moment of inertia in the Y-direction for cylindrical cell units since the ribs between the circular hollow spaces now have a significant influence on the stiffness. An approximate procedure for accounting for this increased stiffness is to replace the cylindrical cell unit with an imaginary prismatic cell unit which has the same first moment of area about the centroid of the section. Thus the diameter of the cylindrical cell, h_i, is replaced by the imaginary height of the equivalent prismatic cell, h_{im}, as follows.

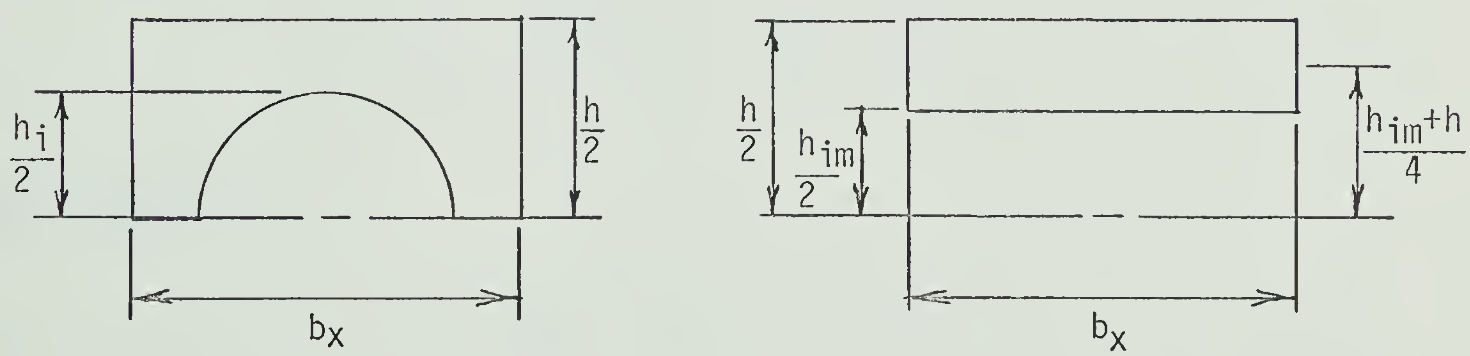

FIGURE B.2 EQUIVALENT CYLINDRICAL AND PRISMATIC CELL UNITS

$$\text{Static Moment of Cylindrical Cell Unit} = \text{Static Moment of Equivalent Prismatic Cell Unit}$$

$$\frac{b_x . h^2}{8} - \frac{\Pi h_i^{\ 2}}{8} \cdot \frac{2h_i}{3\Pi} = \frac{h-h_{im}}{2} \cdot b_x \cdot \frac{h_{im} + h}{4}$$

Therefore,

$$h_{im} = \lambda.h. \sqrt{\frac{2}{3}\,\delta_x} \qquad \text{(B6)}$$

Now using the expression for C_y given for prismatic cells (EQUATION 2.10):

$$C_y = 1 - \delta_y \left(\frac{h_{im}}{h}\right)^3$$

$$= 1 - \delta_y \lambda^3 \left(\frac{2}{3}\,\delta_x\right)^{\frac{3}{2}} \qquad \text{(B7)}$$

From equations (A6) and (B3):

$$J_x = 2I_x = 2.\frac{h^3}{12} \cdot C_x = \frac{h^3}{12} \cdot 2C_{xy}$$

$$J_y = 2I_y = 2.\frac{h^3}{12} \cdot C_y = \frac{h^3}{12} \cdot 2C_{yx}$$

and therefore,

$$C_{xy} = C_x$$

$$C_{yx} = C_y \qquad \text{(B8)}$$

B.3 Prismatic Cells Open at Bottom

This type of slab, which is sometimes referred to as a waffle slab, is shown in FIGURE B.3. The characteristics of the cross-section can again be expressed in terms of:

$$\delta_x = \frac{b_{ix}}{b_x}, \quad \delta_y = \frac{b_{ix}}{b_y}, \qquad \lambda = \frac{h_i}{h} \tag{B9}$$

The bending moment of inertia can be calculated as if the slab was a series of T-beams. Expressed in terms of δ_x, δ_y, and λ the stiffness factors, C_x and C_y, become:

$$C_x = \frac{1 - 4\delta_x\lambda + 6\delta_x\lambda^2 - 4\delta_x\lambda^3 + \delta_x{}^2\lambda^4}{1 - \delta_x\lambda}$$

$$C_y = \frac{1 - 4\delta_y\lambda + 6\delta_y\lambda^2 - 4\delta_x\lambda^3 + \delta_x{}^2\lambda^4}{1 - \delta_y\lambda} \tag{B10}$$

One of Kirchhoff's assumptions for plates is that cross-sections do not warp. However, when a cellular slab has ribs projecting from the surface of the slab, as in FIGURE B.3, it is possible that the ribs may warp under twisting moments. For this condition the torsional constants can be calculated by summing the torsional constant of the slab without ribs with the torsional constants of each rib. The torsional constant for the slab without ribs is based on Kirchhoff's assumption of an unwarped cross-section and the torsional constant for each projecting rib is based on the twisting of a rectangular section free to warp.

Therefore,

$$J_x = \frac{h^3}{12} 2C_{xy} \tag{B11}$$

$$= \frac{1}{b_x} \left(\frac{b_x t^3}{6} + \frac{h_i d_x^{\ 3}}{3} \left(1-0.630 \frac{d_x}{h} + 0.052\left(\frac{d_x}{h}\right)^5\right) \right)$$

Now substituting,

$$t = h\ (1-\lambda) \text{ and } d_x = b_x\ (1-\delta_x)$$

and simplifying,

$$C_{xy} = (1-\lambda)^3 + a_x\ \lambda(1-\delta_x) \tag{B12}$$

where $$a_x = 2\left(\frac{d_x}{h}\right)^2 .\ \left(1\text{-}0630\ \frac{d_x}{h} + 0.052\ \left(\frac{d_x}{h}\right)^5\right)$$

Similarly,

$$C_{yx} = (1-\lambda)^3 + a_y\ \lambda(1-\delta_y) \tag{B13}$$

where $$a_y = 2\left(\frac{d_x}{h}\right)^2 .\ \left(1-0.630\ \frac{d_x}{h} + 0.052\ \left(\frac{d_x}{h}\right)^5\right)$$

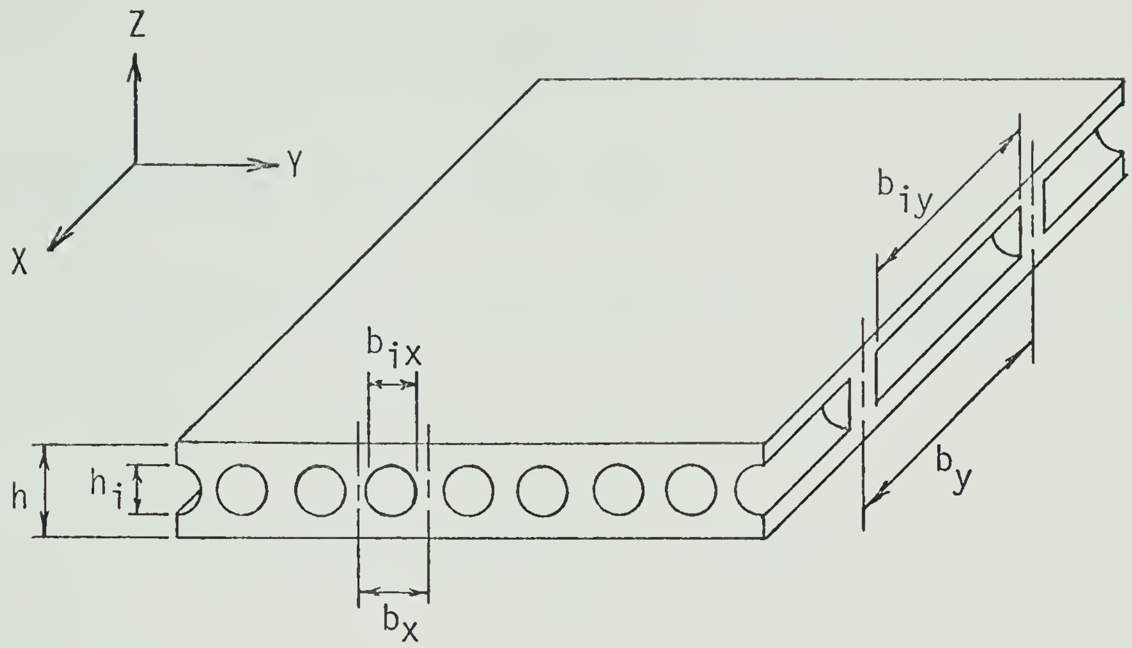

FIGURE B.1 CYLINDRICAL CELL UNITS

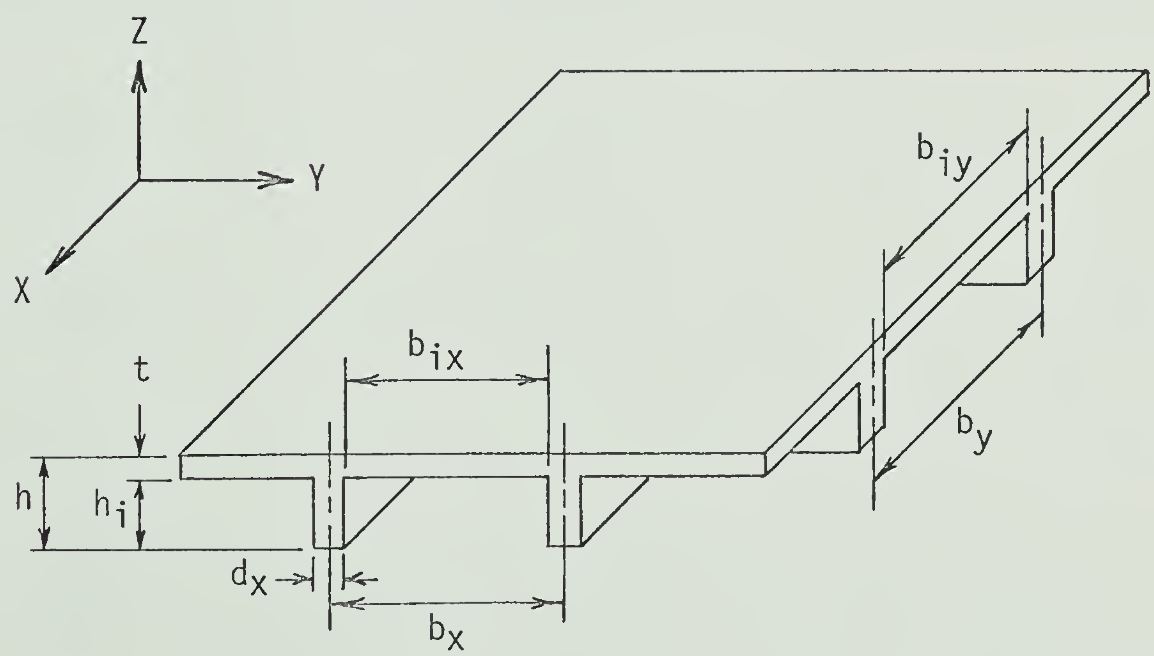

FIGURE B.3 PRISMATIC CELL UNITS OPEN AT BOTTOM

APPENDIX C

FINITE DIFFERENCE PATTERNS

APPENDIX C

FINITE DIFFERENCE PATTERNS

C.1 Difference Pattern for Interior Point

The standard differential equation for an orthotropic plate (EQUATION 2.4) is:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x \partial y} + D_y \frac{\partial^4 w}{\partial y^4} = q \qquad \text{(C1)}$$

The finite difference technique replaces this equation by a series of linear equations at discrete points in the slab. FIGURE C.1 shows a typical internal portion of the slab with a gridwork of points. The point for which the pattern is developed is numbered 7 and the surrounding points are numbered from 1 to 13. The partial differentials of equation (C1) can be replaced by the following difference equations for POINT 7:

$$\left(\frac{\partial^4 w}{\partial x^4}\right) = \frac{1}{hx^4}\left(w_5 - 4\,w_6 + 6w - 4w_8 + w_9\right)$$

$$\left(\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) = \frac{1}{hx^2 hy^2}\left(2w_2 - 4w_3 + 2w_4 - 4w_6 - 8w_7 - 4w_8 + 2w_{10} - 4w_{11} + 2w_{12}\right) \qquad \text{(C2)}$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right) = \frac{1}{hy^4}\left(w_1 - 4w_2 + 6w_7 - 4w_{11} + w_{13}\right)$$

By substituting equations (C2) into equation (C1) we obtain the following equation:

$$
\begin{aligned}
&\frac{D_x}{hx^4}(w_5 + w_6) + \frac{2H}{hx^2hy^2}(w_2 + w_4 + w_{10} + w_{12}) + \frac{D_y}{hy^4}(w_1 + w_{13}) \\
&- \left(\frac{4D_x}{hx^4} + \frac{4H}{hx^2hy^2}\right)(w_6 + w_8) - \left(\frac{4D_y}{hy^4} + \frac{4H}{hx^2hy^2}\right)(w_3 + w_{11}) \\
&+ \left(\frac{6D_x}{hx^4} + \frac{6D_y}{hy^4} + \frac{8H}{hx^2hy^2}\right) w_7 = q
\end{aligned}
\qquad \text{(C3)}
$$

This is the finite difference pattern for a typical internal point.

C.2 Difference Pattern for Point on Free Edge

FIGURE C.2 shows a point on the free edge along the X-axis. If the difference pattern for an interior point (EQUATION C3) is applied to a point on the free edge it will involve four points, 1,2,3, and 4, which are outside the slab and are, therefore, fictitious points. The boundary conditions for a free edge along the X-axis are as follows:

$$
\begin{aligned}
&M_y = D_y \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) = 0 \\
&Q_y - \frac{\partial M_{yx}}{\partial x} = - D_y \frac{\partial^3 w}{\partial y^3} - (2H - \nu D_x) \frac{\partial^3 w}{\partial y \partial x^2} = 0
\end{aligned}
\qquad \text{(C4)}
$$

where Q_y is the shearing force along the free edge. By replacing equations (C4) by difference equations we can write linear equations for the following four conditions:

$$(M_y)_6 = 0$$

$$(M_y)_7 = 0$$

$$(M_y)_8 = 0$$

$$\left(Q_y - \frac{\partial M_{yx}}{\partial x}\right)_7 = 0$$

Using these four equations the fictitious points, 1,2,3 and 4, can be solved for in terms of the real points, 5 to 13. Then by substituting the solutions for the fictitious points into equation (C3) we obtain the finite difference pattern for a point on the free edge along the X-axis involving only real points.

$$\begin{aligned}
&\frac{D_x}{hx^4}(1-\nu^2)\,(w_5 + w_9) - \left(\frac{4D_x}{hx^4}(1-\nu^2) + \frac{2}{hx^2hy^2}(2H - \nu D_y - \nu D_x)\right)(w_6 + w_8) \\
&+ \left(\frac{6D_x}{hx^4}(1-\nu^2) + \frac{2D_y}{hy^4} + \frac{4}{hx^2\,hy^2}(2H - \nu D_y - \nu D_x)\right) w_7 \qquad \text{(C6)} \\
&+ \frac{2}{hx^2hy^2}(2H - \nu D_x)\,(w_{10} + w_{12}) - \left(\frac{4D_y}{hy^4} + \frac{4}{hx^2hy^2}(2H - \nu D_x)\right) w_{11} \\
&+ \frac{2D_y}{hy^4}\, w_{13} = \frac{q}{2}
\end{aligned}$$

The same procedure is used to determine the pattern for a point on the free edge along the Y-axis.

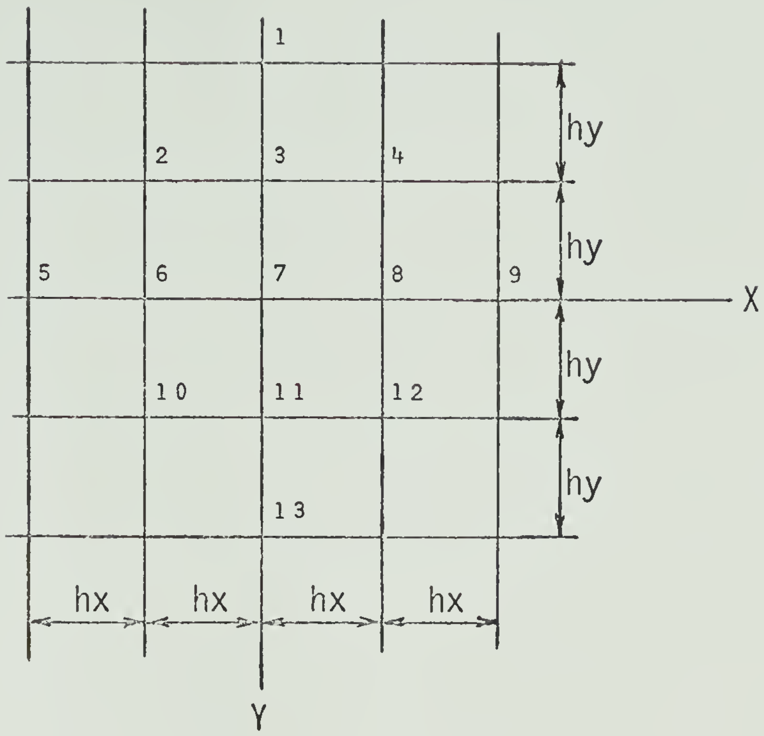

FIGURE C.1 FINITE DIFFERENCE GRID AT INTERNAL POINT

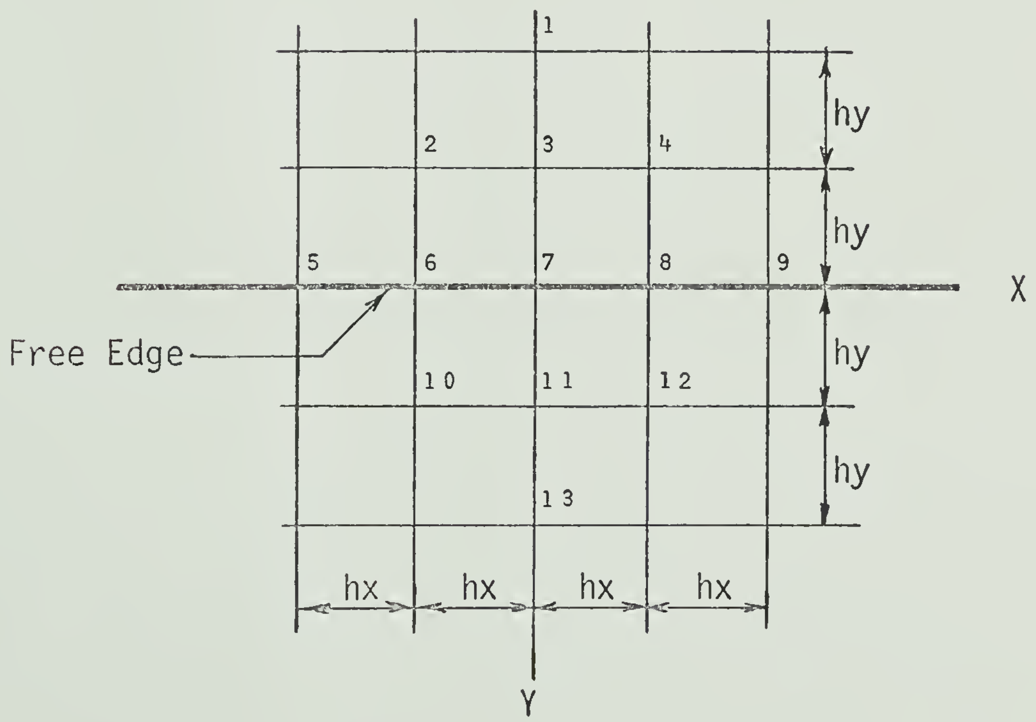

FIGURE C.2 FINITE DIFFERENCE GRID AT FREE EDGE